MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 92 No. 6

MEETING OF 1932 APRIL 8

Dr. H. Knox-Shaw, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Douglas Richard Proctor Coates, 229 Niagara Street, Winnipeg, Manitoba, Canada ;

Henry Rainsford Hulme, B.A., Gonville and Caius College, Cambridge ;

Govind Padmanabha Rao, M.A., Observatory, Survey of India, Dehra Dun, U.P., India ;

Stanley Skewes, M.A., Lecturer in Mathematics, University of Cape Town, S. Africa ;

Rev. W. M. Tate-Stoate, Belle Vue, Blockley, Glos ; and

Alan Herries Wilson, M.A., Emmanuel College, Cambridge.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Ernest William Bliss, Imperial College of Science and Technology, London, S.W. 7 (proposed by E. B. H. Wade) ;

Gustavus Wynne Cook, Wynnewood, Montgomery County, Pa., U.S.A. (proposed by J. Stokley) ;

Victor Dumert, A.M.Inst.B.E., 22 Fortis Green Road, East Finchley, London, N. 2 (proposed by R. C. Morrell) ; and

Leslie Woolven, Langcliffe, 37 Beckway Road, Norbury, Surrey (proposed by A. M. Newbegin).

The following were proposed by the Council as Associates of the Society :—

Professor Giorgio Abetti, Director of the Royal Astrophysical Observatory, Florence, Italy ;

Professor Robert Emden, Professor of Physics, Technical High School, Munich, Germany ;

M. Ernest Esclangon, Director of the Paris and Meudon Observatories, France ;

Professor H. Ludendorff, Director of the Astrophysical Observatory, Potsdam, Germany ; and

Professor P. J. van Rhijn, Director of the Kapteyn Astronomical Laboratory, Groningen, Holland.

Seventy-two presents were announced as having been received since the last meeting, including :—

Athens, National Observatory, Annals, vol. xi;

Cape of Good Hope, Royal Observatory, Annals, vol. x, pt. 9 (The Spectrum of Nova Pictoris, 1925);

C. R. Davidson, "Astrophotographie";

A Marine Chronometer by Arnold (presented by Mr. J. J. Hall).

ON THE IRREGULARITIES OF PERIOD OF LONG-PERIOD VARIABLE STARS.

S. Plakidis, D.Sc.

Introduction.—In a previous paper (*M.N.*, **90**, 65, 1929) I have investigated, in collaboration with Professor Sir A. S. Eddington, the type of irregularity of period of the two well-known long-period variables, o Cet and χ Cyg, specially chosen from among this kind of stars because long series of observations are available. For this investigation an original method of mathematical analysis was applied, and the conclusions arrived at may be summarised as follows :—

I. The irregularity of the period of the above two stars is of the type :

$$\bar{u}_x = \sqrt{(2a^2 + x\epsilon^2)},$$

viz. it consists of purely accidental fluctuations coupled with an a term which probably represents the observational errors in fixing the time of maximum. In other words, there is no evidence of " repeated fluctuations " (positive correlation of the individual periods).

II. The maximum of o Cet being flatter than that of χ Cyg, the constant a for the first star (5·48) is greater than for the second (1·58).

III. The ϵ terms for these two stars (representing irregularities obeying the hypothesis that the individual period differs from the mean period by

a purely accidental fluctuation independent of previous fluctuations) are very nearly in the same proportion as their periods.

Aim of the Present Investigation.—The interesting character of the above results induced the author to proceed to an application of the new method of analysis to a greater number of long-period variables with a view to:

(*a*) Investigating the type of irregularity of their period.

(*b*) Ascertaining whether the above features II and III are purely the results of chance, or constitute general laws holding good for all the long-period variable stars or for a certain group of them.

(*c*) Finding out if the larger value of ϵ is associated with the larger light range rather than with the longer period.

(*d*) Comparing the results of this investigation with those obtained by Dr. W. J. Luyten * in connection with the stars T Gem and R UMa.

Authorities.—For several reasons the observations of the minima of the majority of the long-period variable stars are fewer than those of their maxima. Consequently, as in this investigation it is essential to have available many individual values of the observed period of each star, it has been preferred to make use only of the dates of their maxima—although it would probably be interesting, at least for purpose of comparison, to apply the same method of analysis also in the case in which the mean period is deduced from an extensive series of minima free from long or frequent gaps.

The Julian days of the observed maxima of the stars considered here have been taken from the tables given in vol. lv of the *Annals of the H.C.O.* The tables in question have been extended up to the year 1927 inclusive by the data given in the *Reports of the V.S.S. of the B.A.A.* For a certain number of variable stars the dates of their last maxima were obtained through the courtesy of Mr. Felix de Roy, Director of the V.S.S. of the B.A.A., or deduced from observations published in the *Bulletin of the Observatory of Lyon* and in *Popular Astronomy*.

Wherever in the *Annals of the H.C.O.* more than one date for the same maximum is given, the mean has been adopted.

Stars Examined.—In this paper the 17 long-period variable stars given in Table I are concerned.

The data of that table were taken from the Appendices of the eighth and ninth *Reports of the V.S.S. of the B.A.A.*, and vol. lv of the *Harvard Annals*.

Calculations.—The application of the analysis mentioned above entails calculations which, although very simple, are rather long and tedious unless calculating machines are used.

The greatest part of the calculations for the stars considered here was made with calculating machines (especially with Brunsviga Dupla or Nova and Burroughs Electric, Class II), which Dr. L. J. Comrie, Superintendent of the Nautical Almanac Office, Greenwich, was good enough to place at the disposal of the author during his service in that office.

For the calculation of the uniform ephemeris it has not been considered necessary to proceed to a least-squares solution for a new determination of

* Dr. W. J. Luyten, *Observations of Variable Stars*, Leiden, 1921.

the elements of each star. It has been preferred to take as origin a certain date of observed maximum, and for the constant value of the period a mean value obtained from a series of observations as long continued as possible.

TABLE I

Ref. No.	Star	Period from Max.	Magnitude at		Range	$\frac{M-m}{P}$
			Max.	Min.		
		d.			m.	
1	021024 R Ari	187	8·3	13·4	5·1	0·47
2	050953 R Aur	456	7·9	13·0	5·1	·53
3	074323 T Gem	288	8·1	<13·5	...	·50
4	094211 R Leo	317	6·0	10·2	4·2	·45
5	103769 R UMa	299	7·6	12·9	5·3	·39
6	123760 T ,,	254	7·6	12·8	5·2	·41
7	123961 S ,,	223	7·9	11·5	3·6	·51
8	132422 R Hya	405	4·5	9·5	5·0	·49
9	143227 R Boo	222	7·1	12·2	5·1	·45
10	151731 S CrB	361	7·1	12·9	5·8	·37
11	154615 R Ser	359	6·8	13·2	6·4	·42
12	180531 T Her	165	7·8	12·9	5·1	·48
13	190108 R Aql	314	6·0	11·3	5·3	·42
14	193449 R Cyg	419	7·4	13·8	6·4	·35
15	205923 R Vul	136	8·0	12·6	4·6	·49
16	210868 T Cep	383	6·0	10·2	4·2	·55
17	235350 R Cas	428	6·8	12·9	6·1	·41

TABLE II

Star	$\zeta_0 + PE$	a	ϵ	σ
021024 R Ari	2408450 + 186·5 E	3·65	1·45	6·20
050953 R Aur	2397528 + 287·6	3·54	17·32	13·16
074323 T Gem	2401459 + 457·8	7·34	5·49	12·45
094211 R Leo	2462875 + 313·2	1·58	6·93	9·85
103769 R UMa	2413043 + 301·2	1·58	4·95	9·54
123760 T ,,	2400720 + 256·7	2·24	5·29	10·67
123961 S ,,	2400572 + 225·4	2·24	6·56	12·40
132422 R Hya	2411514 + 423·2	6·00	45·00	20·90
143227 R Boo	2399832 + 223·3	1·57	2·86	6·30
151731 S CrB	2400655 + 361·2	5·74	4·00	11·80
154615 R Ser	2388491 + 357·2	2·74	5·29	10·10
180531 T Her	2403409 + 164·9	3·74	1·41	6·82
190108 R Aql	2399168 + 328·2	2·24	100·00	16·50
193449 R Cyg	2398524 + 425·1	7·07	6·32	17·90
205923 R Vul	2402516 + 136·9	4·12	1·73	6·65
210868 T Cep	2408460 + 387·7	8·51	3·87	14·75
235350 R Cas	2398399 + 431·0	1·00	9·06	11·22

Table II gives the equations of maxima of the variable stars concerned, together with the respective values of the constants a, ϵ, and the mean squares deviation σ, *i.e.* the mean of the squares of the deviations of each period from the mean period.

The above values of the constants a and ϵ of the theoretical curve

$$\bar{u}_x = \sqrt{(2a^2 + x\epsilon^2)}$$

of each star were determined so that the left-hand part of this curve might fit as well as possible with the curve of the observed $\bar{u}_x$.

Table III contains the values of the correlation coefficient $r_x = \frac{\overline{\epsilon_s \epsilon_{s+x}}}{\sigma^2}$ (for different values of $x = 1 \ldots 20$) between (1) consecutive periods; (2) periods separated by 1-period intervals; (3) periods separated by 2-period intervals; and so on.

Diagrams.—Diagrams for each star are appended to this note. They show the fit of the curve of the observed $\bar{u}_x$ (jagged curve) with the theoretical (smooth) curve. They are numbered according to the reference numbers of the stars given in Table I.

Conclusions.—The results obtained from this investigation may be summarised as follows :—

1. None of the 17 stars examined shows any periodic irregularity; and if for some of them the periodogram, or the curve drawn in accordance with the method of analysis applied here, shows some sinuosities, these are believed to be due to the influence either of the error in fixing the dates of maximum or of the gaps in the series of observations, rather than to hint at the existence of some physical cause which might be responsible for a really periodic irregularity.

2. The successive periods of all except four stars are negatively correlated; *i.e.* the irregularities of their periods follow the law of accidental and independent errors, or are combinations of permanent with temporary irregularities.

3. The periods of the excepted four stars, R Aur, R UMa, R Hya and R Aql, are positively correlated, which means that their periods are subject to " repeated " irregularities.

4. The results of this investigation seem to be in favour of the assumption that R Aur belongs to the same category as R Hya and R Aql, the periods of which are well known to decrease.

5. The decrease of period of R Aur and R Hya is continuous, whilst the period of R Aql decreases in part continuously and in part by sudden jumps.

6. With reference to two stars, T Gem and R UMa, which Dr. Luyten says are suspected members of the group referred to in the conclusion 4, it seems that such a suggestion is excluded for the first star; but for the second Luyten's inference is well founded, and the period of the star in question is diminishing by sudden jumps which occur at very long intervals.

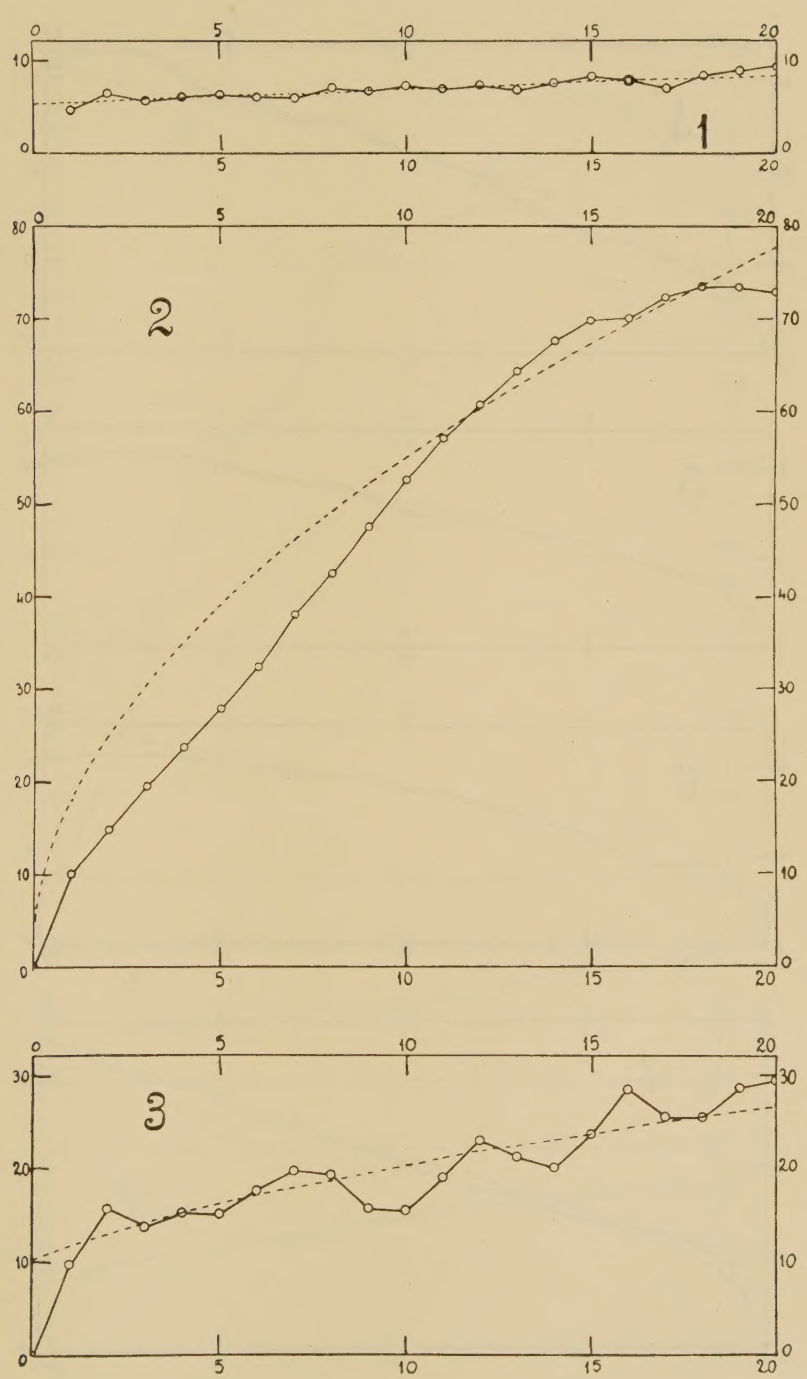
7. As stated above, the constants ϵ of o Cet and χ Cyg were found to be nearly in the same proportion as their periods, viz. the ratio of the mean deviation of an individual period of o Cet to the mean period of this star is

TABLE III

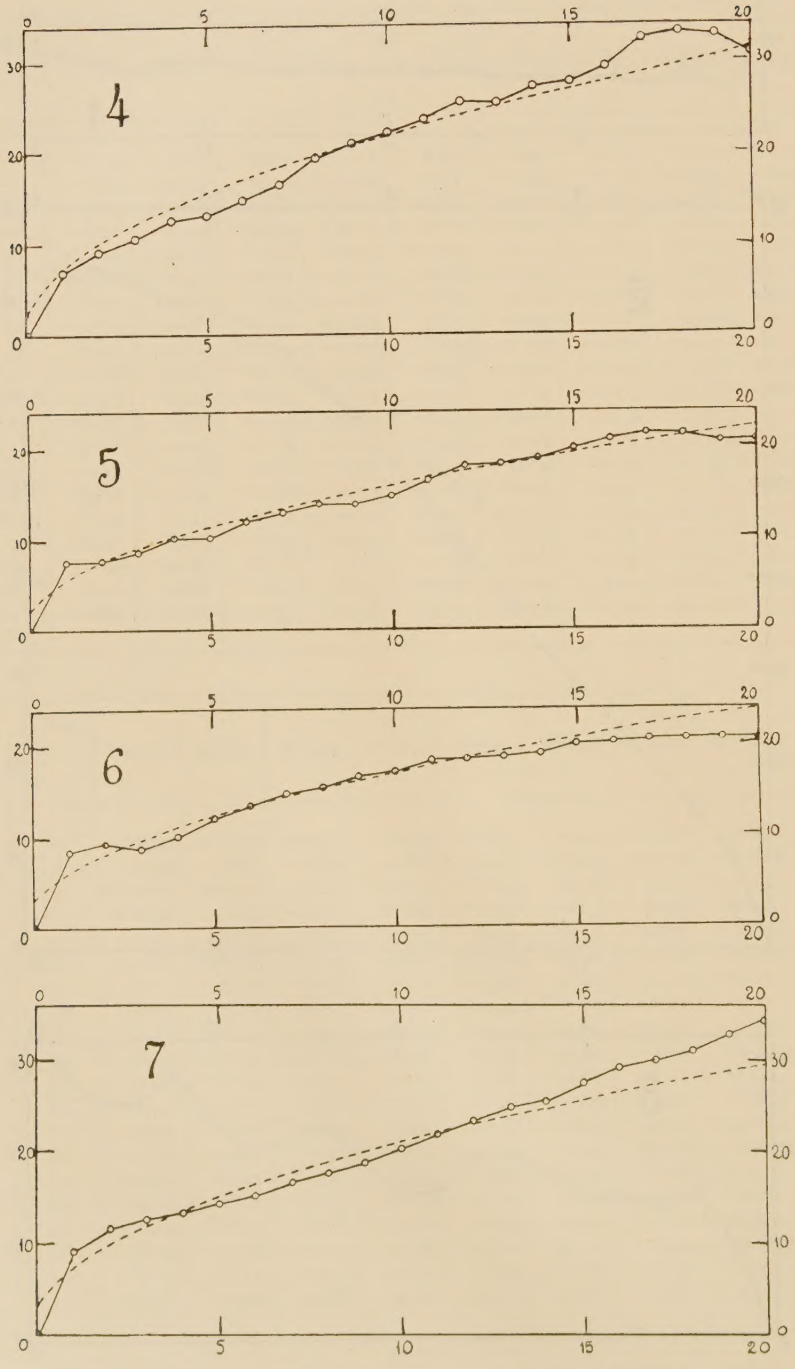
Star / x	R Ari	R Aur	T Gem	R Leo	R UMa	T UMa	S UMa	R Hya	R Boo
1	−0·35	+0·03	−0·17	−0·11	−0·46	−0·45	−0·20	+0·55	−0·46
2	−0·10	+0·25	−0·49	+0·14	+0·35	+0·02	−0·13	+0·59	+0·02
3	+0·05	+0·14	+0·29	+0·05	+0·31	+0·10	−0·15	+0·61	−0·04
4	−0·18	+0·23	−0·29	+0·07	−0·02	+0·22	+0·18	+0·69	+0·17
5	+0·09	+0·29	−0·10	+0·02	+0·03	−0·24	−0·11	+0·68	−0·27
6	+0·26	+0·21	+0·13	+0·11	−0·06	+0·10	+0·14	+0·62	+0·16
7	+0·06	+0·03	−0·14	−0·20	+0·02	−0·02	+0·10	+0·48	+0·07
8	−0·22	+0·06	−0·39	+0·28	0·00	−0·04	−0·10	+0·69	−0·28
9	+0·07	−0·14	+0·34	−0·02	+0·07	−0·10	+0·02	+0·53	+0·15
10	−0·28	+0·21	−0·12	+0·05	−0·06	+0·20	+0·16	+0·53	+0·08
11	+0·53	−0·10	−0·16	+0·27	+0·18	−0·12	−0·11	+0·49	+0·07
12	+0·07	−0·18	+0·25	+0·12	−0·11	−0·25	−0·12	+0·42	−0·01
13	−0·08	−0·25	+0·03	−0·15	+0·02	+0·09	+0·21	+0·37	−0·20
14	+0·13	−0·16	−0·17	+0·06	+0·01	+0·02	−0·01	+0·32	+0·09
15	−0·11	−0·01	−0·18	−0·35	−0·03	−0·06	−0·04	+0·37	0·00
16	−0·42	−0·31	+0·43	+0·15	+0·01	−0·17	−0·11	+0·35	−0·04
17	+0·71	+0·08	−0·19	−0·02	−0·07	+0·10	+0·11	+0·19	−0·15
18	−0·04	−0·44	+0·18	−0·17	−0·14	+0·10	+0·07	+0·23	+0·10
19	−0·22	−0·21	+0·23	−0·03	+0·28	−0·04	+0·03	+0·32	−0·04
20	+0·07	+0·16	−0·77	+0·16	−0·11	−0·08	−0·02	+0·17	−0·32

Star / x	S CrB	R Ser	T Her	R Aql	R Cyg	R Vul	T Cep	R Cas
1	−0·39	−0·37	−0·47	+0·59	−0·62	−0·07	−0·36	−0·16
2	−0·09	−0·05	−0·20	+0·89	+0·23	−0·34	+0·06	+0·08
3	−0·12	+0·10	−0·15	+0·87	−0·14	−0·08	−0·07	+0·10
4	+0·30	−0·05	+0·31	+0·86	+0·34	+0·15	−0·06	−0·01
5	−0·02	−0·21	−0·04	+0·79	−0·43	+0·12	+0·08	−0·17
6	−0·08	−0·16	+0·05	+0·80	+0·49	−0·18	−0·10	+0·11
7	0·00	+0·55	−0·03	+0·71	−0·50	−0·04	+0·20	−0·02
8	−0·02	−0·27	−0·09	+0·58	+0·40	−0·02	−0·33	−0·05
9	+0·13	+0·15	−0·01	+0·77	−0·24	+0·16	+0·29	−0·17
10	−0·13	+0·15	−0·16	+0·48	+0·13	−0·20	+0·05	+0·10
11	−0·06	−0·14	+0·34	+0·67	−0·20	+0·06	−0·13	+0·24
12	+0·11	−0·42	−0·10	+0·53	+0·20	+0·05	+0·11	−0·16
13	−0·07	+0·20	−0·19	+0·72	−0·35	+0·10	−0·25	+0·16
14	+0·06	−0·03	0·00	+0·40	+0·52	+0·01	+0·11	+0·03
15	−0·03	−0·04	+0·03	+0·59	−0·62	−0·07	+0·06	−0·22
16	+0·01	+0·16	+0·12	+0·40	+0·25	+0·06	−0·06	−0·09
17	−0·05	+0·21	−0·08	+0·52	−0·09	−0·02	−0·13	−0·12
18	+0·02	+0·18	+0·22	+0·60	+0·26	+0·11	−0·07	−0·10
19	−0·20	−0·11	−0·12	+0·42	−0·24	+0·07	+0·25	−0·03
20	−0·31	−0·09	−0·18	+0·28	+0·47	−0·08	−0·19	−0·35

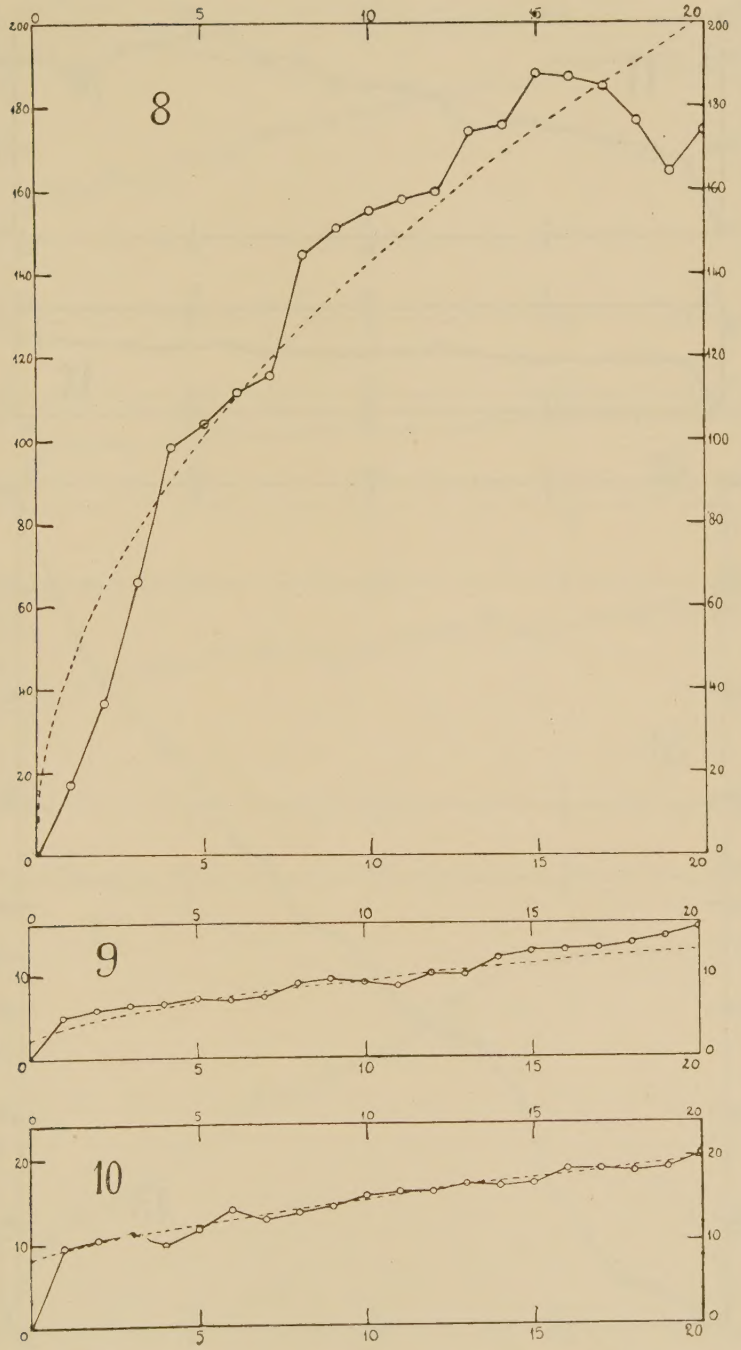
1·35 per cent., while that of χ Cyg is 1·39 per cent. An inspection of the following Table IV leads to the conclusion that the constancy of the ratio $100\epsilon/P$ observed in these two stars is not a matter of coincidence. Except



1. *R Arietis.* 2. *R Aurigæ.*
3. *T Geminorum.*

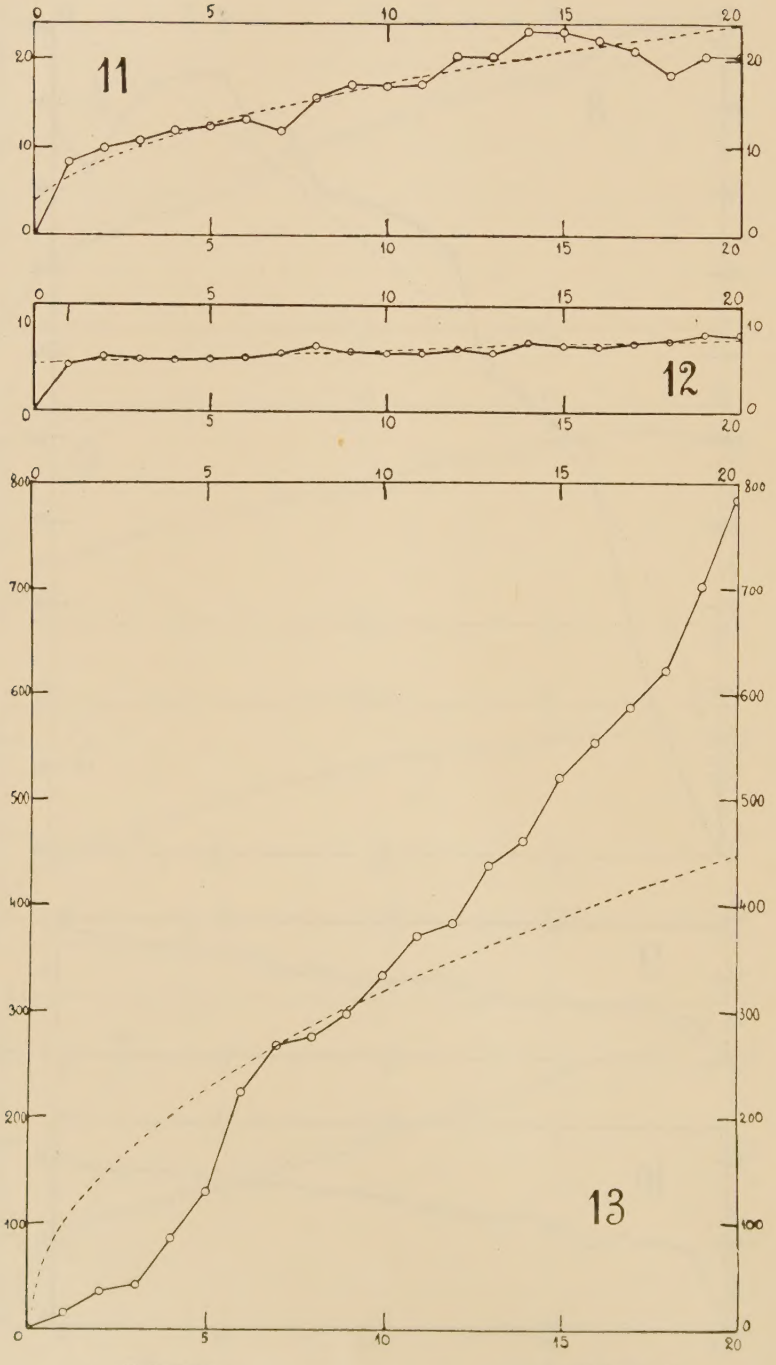


4. *R Leonis.*
5. *RU. Majoris*
6. *TU. Majoris.*
7. *SU. Majoris.*



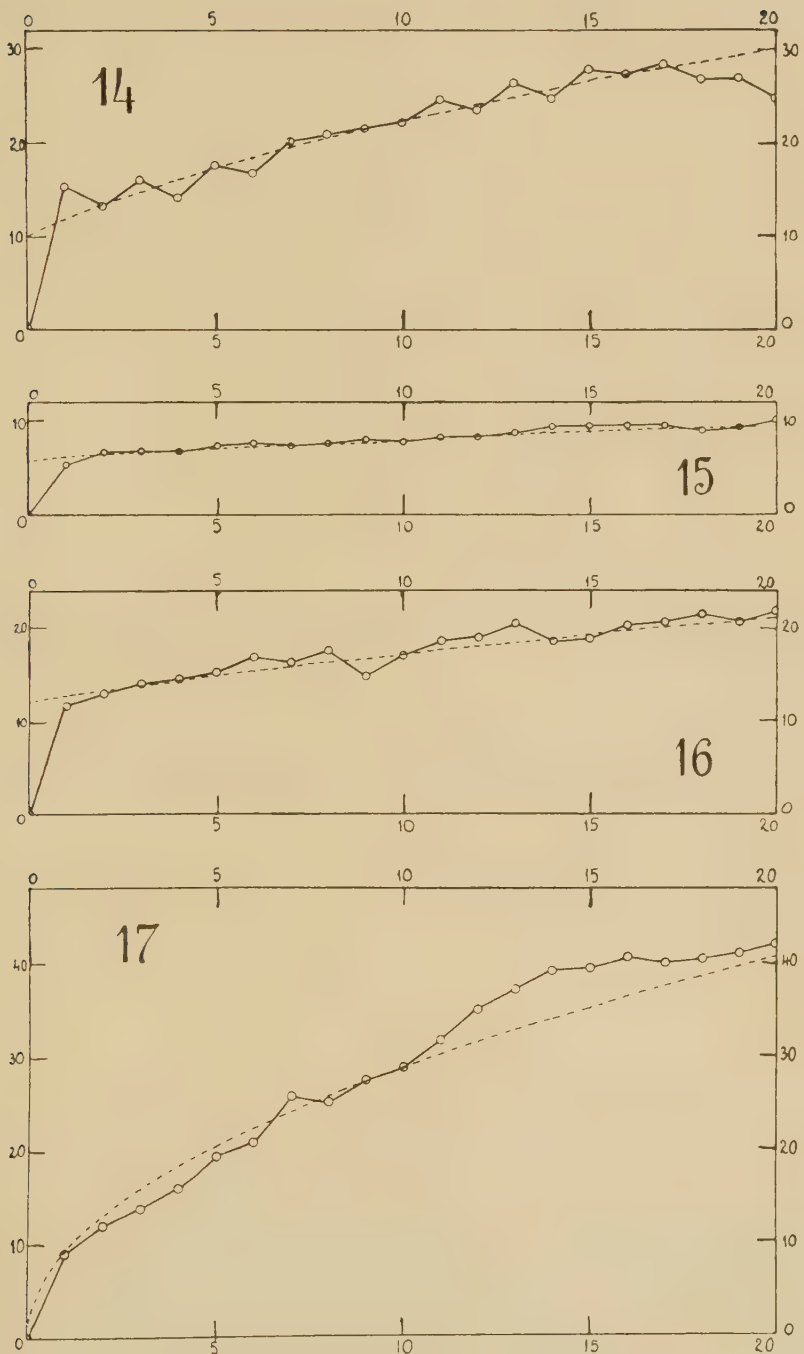
8. *R Hydræ.* 9. *R Bootis.*

10. *SCor. Borealis.*



11. *R Serpentis.* 12. *T Herculis.*

13. *R Aquilæ.*



14. *R Cygni.*
15. *R Vulpeculæ.*
16. *T Cephei.*
17. *R Cassiopeiæ.*

for the three stars of conclusion 4, and two more, T Gem and S UMa, for which the values of this ratio are a little larger, the values of ϵ show a tendency to increase with the period *P*.

TABLE IV

Star	$\frac{100\epsilon}{P}$	Range	Star	$\frac{100\epsilon}{P}$	Range
		m			m
R Ari	0·77	5·1	S CrB	1·11	5·8
R Aur	3·78	5·1	R Ser	1·48	6·4
T Gem	4·33	6·1	T Her	0·85	5·1
R Leo	2·22	4·2	R Aql	3·05	5·3
R UMa	1·65	5·3	R Cyg	1·49	6·4
T ,,	2·06	5·2	R Vul	1·27	4·6
S ,,	2·92	3·6	T Cep	1·00	4·2
R Hya	4·94	5·0	R Cas	2·10	6·1
R Boo	1·28	5·1			

8. The constant *a* has almost always a high value if the light-curve of the star is flattened round its maximum. This confirms our view that *a* depends on the accidental errors committed in the determination of the date of maximum.

9. There is no relation between the constant *a* and the period *P*.

10. The search for a relation between the constant *a* and the range of light fluctuation was also one of the objects of the present investigation. There is no clear indication of any connection.

I wish to express my thanks to Professor Sir A. S. Eddington for his interest in this research.

NOTE ON DR. PLAKIDIS'S PAPER.

Sir A. S. Eddington, F.R.S.

Having been associated with Dr. Plakidis in the first paper on this subject, I may conveniently state here my general conclusions from the greatly extended observational material which he has now made available. Including *o* Cet and χ Cyg already treated, the investigation covers 19 stars. Among these there are several stars which show abnormal effects, notably R Aql (fig. 13). This has a decreasing period (according to Plakidis, decreasing by sudden jumps); such a change comes under our heading "repeated irregularities," *i.e.* a deviation of period repeated in a number of successive periods. In the curves it manifests itself by $\bar{u}_x$ tending to vary as x instead of $\sqrt{x}$; and in the table of correlation coefficients it is shown by positive coefficients. Similar changes are shown by R Aur and R Hya (figs. 2, 8).

It must be remembered that the left-hand parts of the curves have much greater weight than the right-hand parts, so that the deviation of the curves for these two stars at the extreme left is highly significant. It is confirmed by the positive coefficients in Table III.

Setting aside these and two or three other stars mentioned by Plakidis as having minor abnormalities, we have a group consisting of about three-quarters of the whole number which follow a normal pattern. In these the whole irregularity appears to consist of a purely accidental fluctuation of period with a purely accidental fluctuation of phase superposed. The latter is presumably the accidental error in observing the time of maximum ; for, as Plakidis points out, it is largest in the stars with flat maxima for which it is most difficult to fix the exact date of maximum. It is the quantity a in Table II, and ranges from 1 to 8 days. If maximum light is not an exact criterion of the phase of the more deep-seated pulsatory change going on in the interior of the star, the accidental difference will also appear as part of a. This term is responsible for the negative correlation coefficients in the first line of Table III ; for an error in observing the time of maximum lengthens one period and shortens the next, or *vice versa*.

Removing the a term, we are left with an accidental fluctuation of period whose mean value is denoted by $\pm \epsilon$. The first two stars examined suggested that the percentage fluctuation of period might be constant, viz. $\epsilon = 1{\cdot}4$ per cent. of the period. Plakidis finds that ϵ tends to increase with the period, but the exact agreement of the ratio for o Cet and χ Cyg seems to have been accidental. Nevertheless there is considerable uniformity ; for 14 stars, ϵ ranges from 0·77 to 2·22 per cent. of the period. Of the remaining 5 stars the 3 with definitely changing periods must necessarily be excluded, since the calculated ϵ is meaningless for them ; there remain 2 stars (S UMa and T Gem) which have exceptionally large fluctuations but are not otherwise abnormal.

I conclude therefore that, whilst exceptional stars occur, nearly three-quarters of the long-period variables constitute a normal class in which successive periods fluctuate accidentally, the mean deviation from the mean period being from 1 to 2 per cent.

THE HYDROGEN CONTENT OF THE STARS.

Sir A. S. Eddington, F.R.S.

1. When the luminosity of a star is computed from its mass and radius with the value of the absorption coefficient obtained from pure physical theory, the result comes out too bright. This well-known discrepancy amounts to a factor 10 for diffuse or massive stars, and is still larger for the Sun and smaller stars in a less highly ionised condition. This result is subject to the reservation that the stars do not contain a large proportion of hydrogen. When I first found the discrepancy I showed that the calculated and observed luminosities would agree (in the standard star Capella) if the

material contained 20 per cent. of hydrogen by weight ; * but I did not at that time think that abundance of hydrogen was the actual explanation of the discrepancy.

Partly by elimination of alternative explanations, and partly by the recent evidence of great abundance of hydrogen in stellar atmospheres coupled with our theoretical knowledge that hydrogen will not escape to the outside but will be kept stirred by currents set up in rotating stars, the hydrogen hypothesis has now come into prominence. If the proportion of hydrogen is given, the composition of the rest of the material makes very little difference to the luminosity. Thus if we are convinced that there is no other serious uncertainty in the problem, it is theoretically possible to determine the hydrogen content of a star of known L, M and R. We have simply to find what proportion of hydrogen mixed with other elements will give a luminosity agreeing with the observed luminosity. This is no longer a matter of speculative curiosity ; such determinations are needed to compare with and check the determinations of abundance of hydrogen in stellar atmospheres made by H. N. Russell and others.

The present paper contains the results of such determinations. I have used the new theory and table of the " guillotine factors," † which was prepared to facilitate this investigation.

2. For each star there are two solutions—two possible proportions of hydrogen consistent with the observed luminosity. In one solution the star is chiefly hydrogen (about 99½ per cent.) with only a trace of other elements. The other solution, which rightly or wrongly I have assumed to be the more probable, gives approximately 33 per cent. hydrogen in the Sun, Capella, Algol and Krueger 60. These stars were selected as having first-class observational data and covering a wide variety of mass and density. There is some indication that the most massive stars have a larger proportion of hydrogen, but it is rather dubious.

The luminosity is decidedly sensitive to the assumed proportion of hydrogen, so that if there is no unrecognised complication a useful determination should be obtained. The surprising thing is the steadiness of the hydrogen content, shown not only in the four stars above mentioned, but in the general adherence of the stars to a mass-luminosity curve. If stars of the same mass differed considerably in hydrogen content, deviations from the mass-luminosity curve would be much wider and commoner than we observe. The luminosity of a star having the mass and radius of the Sun can range over 7 magnitudes according to the proportion of hydrogen. It will be hard to reconcile any theory of the progressive evolution of complex elements out of hydrogen in the interior of a star with this result.

The discovery of a new element, neutron, which might well be of fundamental cosmical importance, was announced whilst this investigation was in progress. Neutron should have molecular weight 1, which is very near to the mean value for the mixture of 33 per cent. hydrogen with other elements used in this investigation. Thus even if neutron were very abundant it would not affect the molecular weight and the temperatures

* *M.N.*, 84, 114, 1924. † *Ibid.*, 92, 364, 1932.

here derived. Its only effect would be to reduce the opacity and increase the luminosity by the proportion of the mass representing idle neutron. For example, if 50 per cent. of the whole mass were neutron the luminosity would be doubled ; and to compensate this in the Sun it would be necessary to raise the proportion of hydrogen in the remaining half of the mass from 33 to 40 per cent. (*i.e.* 20 per cent. of the whole mass).

After my calculations (except for Algol and the Cepheids) had been completed, I received a paper by B. Strömgren * dealing with the same problem. Our results are in remarkably close agreement, and many of the points brought out in this paper appear also in his. The principal difference in our treatment is that he adopts the mixture of elements found by Russell in the solar atmosphere and calculates its opacity (including guillotine factors) directly, whereas I employ a general theory of the guillotine factor which should apply to any likely mixture. The accordance of the results supplies a valuable check.

I have based the calculations on Kramers's theory of absorption, understanding that this was closely confirmed by the modern results of wave-mechanics. I have since looked into the comparison more closely and found the agreement is not so good as I was led to believe. There has been some confusion owing to the faulty weights of states in use when Kramers's investigation was made, so that it is impossible for his emission formulæ and absorption formulæ both to be right as they stand. As explained in § 9, wave-mechanics would give a somewhat higher proportion of hydrogen, about 40 per cent. instead of 33 per cent.

The calculations depend on the theory given in *Internal Constitution of the Stars* (quoted as *I.C.S.*). The following example will show the procedure :—

Example of the Calculation

3. We calculate the total radiation of the Sun from the data, $M = 1{\cdot}98 \,.\, 10^{33}$ gm., $R = 6{\cdot}95 \,.\, 10^{10}$ cm. on the hypothesis that it consists of $33\frac{1}{3}$ per cent. H and $66\frac{2}{3}$ per cent. $\overline{Fe}$. ($\overline{Fe}$ denotes a mixture of elements with iron as the mean.)

Accordingly a total weight equal to 84 hydrogen atoms is represented by 1 $\overline{Fe}$ atom of weight 56 and 28 hydrogen atoms. Preliminary calculation indicates that at the mean temperature and density of the star, $\overline{Fe}$ retains about 4 electrons. We have therefore

$$\begin{aligned} 1\ \overline{Fe} &= 1 \text{ ion} + 22 \text{ free electrons}, \\ 28\ H &= 28 \text{ ions} + 28 \text{ free electrons}, \\ \text{Total} &= 29 \text{ ions} + 50 \text{ free electrons} = 79 \text{ molecules}. \end{aligned}$$

The molecular weight $\mu = 84/79 = 1{\cdot}063$.

Then β is calculated from the usual quartic equation

$$1 - \beta = {\cdot}00309(M/\odot)^2\mu^4\beta^4, \qquad (1)$$

* *Zeits. für Astrophysik*, 4, 118 (March 1932). His introductory section gives an admirable review of the history of the problem.

which gives

$$1-\beta=\cdot00389.$$

The central temperature in the model $n=3$ is (*I.C.S.* equation (99·3)),

$$T_c=0\cdot856\frac{G}{\mathfrak{R}}\frac{\mu\beta M}{R}=20\cdot9\,.\,10^6. \tag{2}$$

The central density is

$$\rho_c=54\cdot36\rho_m=76\cdot7. \tag{3}$$

The number of free electrons per c.c. at density ρ_c is

$$\sigma_0=\frac{50}{84}\frac{\rho}{\mathrm{H}}=2\cdot75\,.\,10^{25},$$

H being the mass of a hydrogen atom. The dominant ionisation potential ψ_1 is given by

$$\psi_1/R'T=\log_e\left\{\tfrac{1}{2}\sigma_0\left(\frac{h^2}{2\pi mR'T}\right)^{\frac{3}{2}}\right\}=-2\cdot829 \text{ at the centre,} \tag{4}$$

where Boltzmann's constant is denoted by R' to distinguish it from the radius R.

The mean conditions in the star correspond to a point where T is about $\frac{2}{3}T_c$ (cf. *I.C.S.*, §§ 62, 94). To reduce $\psi_1/R'T$ to this point we must add $\frac{3}{2}\log_e\frac{3}{2}=\cdot608$. Thus $\psi_1/R'T=-2\cdot221$. The corresponding guillotine factor according to my new table * is $g=5\cdot80$.

The opacity k_c corresponding to T_c and ρ_c, calculated without the guillotine factor, is †

$$k_c=[25\cdot2191]\frac{Z^2}{A}\frac{\rho_c}{\mu T_c^{\frac{7}{2}}(1+f)}. \tag{5}$$

For iron we take the effective charge Z to be 24, as explained in § 4. A must be taken to be 84, *i.e.* the total weight per iron atom, not merely the weight of the iron atom. The absorption performed by the hydrogen ions is negligible. The result is $k_c=124\cdot5$, $k_c/g=21\cdot46$.

Finally,

$$L=\frac{4\pi cGM(1-\beta)}{\alpha k_c/g}, \tag{6}$$

where α is between 1·32 (uniform distribution of energy-source) and 4·25 (point-source). We adopt $\alpha=2\cdot5$ as the usual standard value. The result is $L=3\cdot61\,.\,10^{33}$, which differs only by $0^{\mathrm{m}}\cdot05$ from the observed value, $3\cdot78\,.\,10^{33}$.

4. The mean conditions correspond to a temperature $T=\frac{2}{3}T_c=13{,}900{,}000$ degrees, and $R'T=1200$ electron volts. Thus the dominant ionisation potential ψ_1 is 2670 volts, which is between the K and L levels for iron. There will be practically no ionisation of the K electrons at 9000 volts; the ionisation of the L electrons at 2000–1100 volts will be incomplete, and I have therefore allowed for two L electrons being retained. It may be

* *M.N.*, 92, 364, 1932. † *I.C.S.*, equation (158·1).

noticed that an error in this estimate is less serious than it would be in a hydrogenless star, since the majority of the free electrons are provided by the fully ionised hydrogen.

According to the table of guillotine factors (*loc. cit.*), most of the weight of the opacity coefficient depends on radiation of energy between $3{\cdot}8\ R'T$ and $8{\cdot}9\ R'T$. Deducting $R'T$ for the positive energy of the electrons before capture, this gives capture levels from 3400 to 9500 volts. For these captures the effective charge consists of the nucleus and the two K electrons ; I have accordingly adopted $Z = 26 - 2 = 24$. The disturbing effect of the electrons in orbits above the capture level has not been investigated theoretically, but it is presumed that it is negligible ; for otherwise Kramers's theory of absorption would not give correct results when applied, as it has been in practice, to complete atoms. (His theory deals with the field due to a simple nuclear charge, but it is verified by experimental determinations of the absorption of X-rays by atoms.)

In each of the calculations of which the results are given below, the conditions at temperature $\frac{2}{3}T_c$ have been examined, and the values of μ and Z have been adopted so as to correspond. It turns out that $Z = 24$ applies to all the examples of iron-hydrogen stars considered.

The Sun

5. The following table gives the results obtained for the Sun for different proportions of hydrogen. The first column gives the assumed percentage (by weight) of H, the remainder being $\overline{Fe}$. The second column gives the mean molecular weight. The third column gives the central temperature for the model $n = 3$. The fourth column gives $-\psi_1/RT$ at temperature $\frac{2}{3}T_c$. The fifth column gives k_c/g. The sixth column gives the decimal logarithm of the calculated radiation ; the observed value is $33{\cdot}5775$, and the difference converted into magnitudes is given in the seventh column.

TABLE I

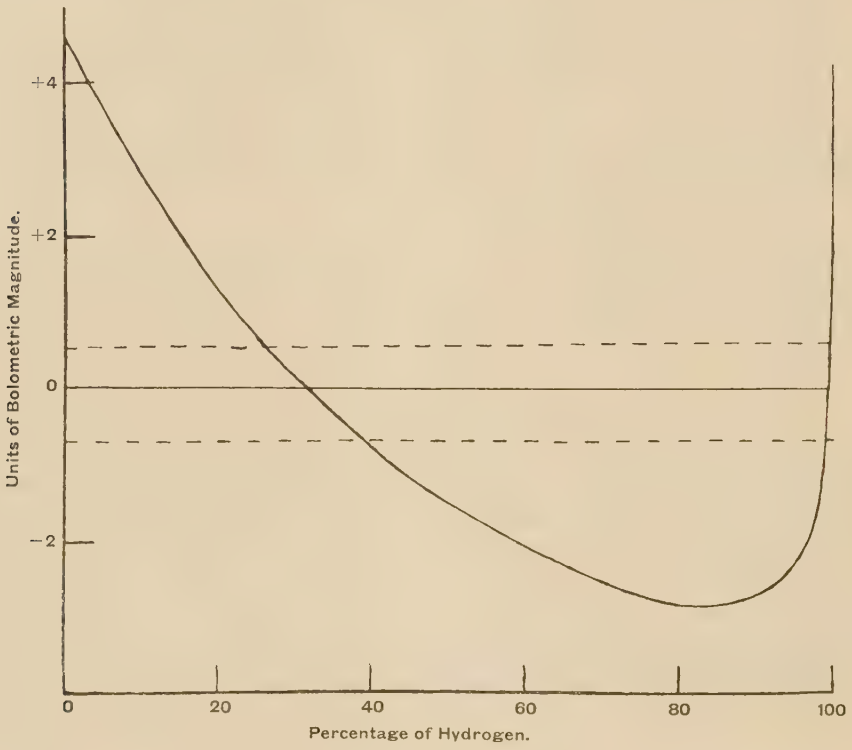
H per cent.	μ	$10^{-6}T_c$	$-\psi_1/RT$	k_c/g	$\log_{10} L$	Residual *
						m
0	2·24	41·5	3·58	4·81	35·3998	+4·56
25	1·21	23·7	2·47	17·24	33·8721	+0·74
33⅓	1·063	20·9	2·22	21·46	33·5579	−0·05
50	0·836	16·5	1·72	29·88	33·0000	−1·44
75	0·631	12·4	1·11	30·21	32·5086	−2·67
90	0·546	10·8	0·78	17·26	32·5020	−2·69
99	0·500	9·9	0·58	2·14	33·2540	−0·81

The residuals are plotted in the diagram. The two points at which the curve crosses the zero-line give two possible compositions of the Sun for which the observed and calculated luminosities will agree. These are found to be 33 per cent. and 99·53 per cent. hydrogen. The existence of the

* A positive residual means that the calculated value of L is too bright.

second solution has been noticed by Lönnqvist, Menzel and B. Strömgren. The first solution, 33 per cent., agrees exactly with Strömgren's calculation.

In addition to the standard zero-line for $\alpha = 2{\cdot}5$, two other zero-lines are shown (dotted) corresponding to the extreme limits, $\alpha = 1{\cdot}32$ and $4{\cdot}25$, in order to indicate the greatest possible effect of uncertainty as to the distribution of the energy-source. The upper dotted line corresponds to a point-source ; this, however, is an ideal limit and not an actually possible distribution, and it is unlikely that the zero-line could be displaced much



Theoretical Constitution of the Sun.

above the full-line. The lower dotted line corresponds to uniform distribution of the source.*

If the hydrogen content of the stars varied arbitrarily from 0 to 70 per cent., stars of the same mass and radius as the Sun would have a range of more than 7 magnitudes. Since the observed range is comparatively small, there must be some cause which makes the hydrogen content of the stars nearly uniform.

6. In order to test the effect of substituting another mean element for iron I have calculated L for the Sun with an assumed composition

* Strömgren obtains the interesting result that the effect of uncertainty of the source distribution is actually rather less than that indicated in my diagram, owing to a compensating effect which I here neglect. He points out that in the point-source model the guillotine factor is rather larger than in the standard model.

32·5 per cent. H and 67·5 per cent. $\overline{Al}$. The figures (corresponding to Table I) are :

32·5 1·000 19·7 2·04 15·7 33·5875 $+0^m\cdot02$

Interpolating in Table I for percentage 32·5, we see that the residual for $\overline{Fe}$ is also $+0^m\cdot02$; so that the substitution of aluminium (atomic number 13) for iron (26) has not altered the calculated luminosity by as much as $0^m\cdot01$. So close an agreement may be exceptional, but it illustrates the fact that the luminosity is in general very insensitive to changes in the assumed composition except as regards hydrogen content.

Strömgren, using " Russell's mixture," adopts $Z^2/A = 6\cdot0$; which is very nearly the value for $\overline{Al}$. The general agreement of his results for $\overline{Al}$ with mine for $\overline{Fe}$ is additional evidence of the insensitivity.

Other Stars

7. Adopting $33\frac{1}{3}$ per cent. hydrogen, I find the following results for other typical stars, the results for the Sun being added for comparison :—

TABLE II

	μ	$10^{-6}T_e$	$-\psi_1/RT$	k_c/g	$\log_{10} L$	Residual
						m
Capella	1·04	6·66	6·54	14·45	35·4661	−0·28
Krueger 60	1·15	21·1	−0·07	33·9	31·7962	+0·75
Algol	1·04	26·6	4·51	4·60	35·9958	+0·01
V Puppis	1·04	35·9	6·12	0·73	38·0993	+1·70
Sun	1·06	20·9	2·22	21·46	33·5579	−0·05

I have adopted the same data for Capella, Krueger 60 and V Puppis as in *I.C.S.*, pp. 145–151, except that in accordance with recent determinations I now take the Sun's absolute bolometric magnitude to be $4^m\cdot60$ instead of $4^m\cdot85$; this change does not affect calculations for the Sun, but it affects the conversion into absolute units of the data for Capella and Krueger 60.* I adopt 0·27 as the mass of the bright component of Krueger 60, leaving 0·18 for the faint component ; this partition is estimated from their difference of magnitude, but I could not follow strictly the method used in *I.C.S.*, since the faint component is too degenerate to be treated by the present formulæ. A smaller residual would have been obtained by using the mass 0·25 currently quoted, but I do not think this value has much weight. For Algol I adopt $M = 4\cdot30\odot$, $R = 2\cdot18 \cdot 10^{11}$, $T_e = 13{,}000°$ (*I.C.S.*, p. 210, footnote).

Capella.—The residual is negligible, so that practically the same proportion of hydrogen is found in the Sun and Capella. Zero residual would be given by assuming 30 per cent. hydrogen. Strömgren's result is 31 per cent.

* But not the eclipsing stars V Puppis and Algol, whose radii are found directly from the orbital data.

The second solution for Capella gives 99·75 per cent. hydrogen, compared with 99·53 per cent. for the Sun.

Krueger 60.—The dominant ionisation potential is practically zero, so that even the hydrogen is only half ionised. Iron retains at least seven of the eight L electrons and about half of the higher electrons.

This retention of weakly bound electrons raises an interesting question of theory. By the correspondence principle there is no discontinuity in the behaviour of electrons at zero energy, and those near the border-line can be treated equally well by the theory of bound or of free electrons. If, treating an M electron as bound, we omit it in calculating the emission by free-bound transitions, we shall have to amend our formulæ to include its line emission in the bound-bound transition to an unoccupied L level. Similarly, if we omit its kinetic energy in calculating the pressure, we must include in the pressure a volume-term on account of its extension of the volume of the ion. It seems clear that when we are dealing with pressures and radiative processes corresponding mainly to energy of order 2000 volts, we can simplify the problem and avoid a number of complicated corrections by ignoring weak binding at 200 or 300 volts ; and *a fortiori* the binding of the hydrogen atoms at 13 volts can be ignored. I have therefore treated the iron as retaining only the ten K and L electrons, and have regarded all the hydrogen electrons as practically free.

There is considerable degeneracy of the free electrons. Thus at the mean energy level $\frac{3}{2}RT$ about 0·18 of the phase-cells are occupied. I have made no allowance for this in the calculation. The residual $0^{m}\cdot75$, by which the observed brightness is less than the calculated brightness, may well be attributed to the degeneracy (*cf.* the low luminosity of white dwarfs). So far as I can judge, the degree of degeneracy is of the right order of magnitude to produce such a residual. If the residual were attributed to a difference of hydrogen content it would mean that Kreuger 60 contains rather more hydrogen than the Sun, which seems unlikely.

The bright component of Krueger 60 is barely within range of our theory. The faint component is much more degenerate and cannot be treated at present.

Algol.—Strömgren states that he finds a residual of over a magnitude, but does not give his adopted data. My own result indicates complete agreement with the theoretical value.

V Puppis.—Owing to the low value of k_c/g it is not permissible to neglect the ordinary scattering by electrons independent of their encounters with ions. The additional opacity coefficient is 0·25, and this has been included in my calculation of L. This complication was also noticed by Strömgren.

The observational data for V Puppis are not especially accurate, and the rather large residual is not in itself important. But the calculation was made in order to examine how the new mass-luminosity curve (for $33\frac{1}{3}$ per cent. hydrogen) deviates from my original curve for masses greater than Capella. The residual for V Puppis (*I.C.S.*, p. 148) was originally $+0^{m}\cdot31$; thus at mass 19 the new curve is running $1^{m}\cdot4$ above the old curve.

This is not a very great change considering that it refers to the extreme part of the curve, where we have very little observational information to check its course. The observational evidence, for what it is worth, is against raising the original curve, which seems already to be rather too high if anything. Apparently therefore we must postulate a larger proportion of hydrogen in the massive stars so as to reduce their luminosities.

There is thus some ground for thinking that the proportion of hydrogen in the most massive stars is greater than 33 per cent. It would be interesting to know whether their atmospheres give a similar indication of excess of hydrogen. If so, it would be in keeping with a view that I have previously expressed : " It is just possible that the occasional stars of exceptionally great mass are those formed from material which happened to be rich in hydrogen." *

But the evidence is rather weak. The observational material consists entirely of eclipsing and Cepheid variables. The latter carry practically no weight, because the results are sensitive to changes in the adopted model ; if they agree, well and good, but if not, there are alternative explanations of the discrepancy. The results for eclipsing variables depend (more sensitively than for ordinary stars) on the assumed effective temperatures, which for these B-type stars are notoriously uncertain. We have assumed that the type B1 of V Puppis corresponds to 19,000° ; if it were 28,000° the residual would be zero.

Cepheid Variables

8. It is of interest to re-examine the theoretical periods of pulsation of Cepheid variables, assuming the composition $\frac{1}{3}H$ and $\frac{2}{3}\overline{Fe}$. I shall use the newer period-luminosity curve and period-spectrum curve given by Shapley, so that changes from my former results (*I.C.S.*, § 130) are due partly to the revised observational data as well as to the change in the adopted composition.

Instead of treating individual Cepheids, I consider four representative stars which obey exactly Shapley's empirical curves, viz. the photographic period-luminosity curve (*Star Clusters*, p. 135), and the period-spectrum curve (*ibid.*, p. 137). For the corresponding colour-index and effective temperature I use Russell, Dugan and Stewart, *Astronomy*, p. 734.

Since these are highly ionised diffuse stars the molecular weight is 1·04, as in Capella, and the guillotine factor is negligible. Using the two empirical curves we find the observed T_e corresponding to any chosen L ; and it is then possible to derive the mass, density, etc., by the theory used in the foregoing part of this paper.

No change is needed in the formula giving the theoretical period of pulsation (*I.C.S.*, § 130), viz. :

$$\Pi\sqrt{\rho_c} = 0{\cdot}290(\gamma\alpha)^{-\frac{1}{2}} = 0{\cdot}290(3\gamma - 4)^{-\frac{1}{2}} ;$$

but the introduction of hydrogen reduces $1 - \beta$ and thereby increases γ, so that the theoretical $\Pi\sqrt{\rho_c}$ is smaller than in my former determinations.

* *I.C.S.*, p. 18.

The following results are obtained for four representative stars. The adiabatic constant Γ of the material is taken to be $\frac{5}{3}$:—

TABLE III

Mass (Sun = 1)	5	8	12	18
Observed period	1·42	7·09	18·76	38·84 days
Calculated period	0·564	3·01	9·57	24·45 ,,
Ratio, O/C	2·51	2·36	1·96	1·59
Abs. bol. mag.	−1·00	−2·58	−3·81	−4·92
T_e	6500	5100	4430	4030
ϱ_c	0·347	0·0147	0·00174	0·000323
$10^{-6}T_c$	9·22	4·10	2·41	1·61
$1-\beta$	·0675	·1308	·2053	·2916
$\Pi\sqrt{\varrho_c}$ (observed)	·834	·859	·783	·698
$\Pi\sqrt{\varrho_c}$ (calculated)	·332	·365	·399	·439

The theory of this computation may be understood by considering the first star, viz. a star with an observed period of 1·42 days. By Shapley's curves we find that (observationally) such a star has effective temperature 6500° and absolute bolometric magnitude $-1^{m}\cdot00$. The radius (not given in the table) is then immediately obtained, viz. $7\cdot19 \cdot 10^{11}$.

We now apply the theory of the mass-luminosity relation (as in preceding sections of this paper) to deduce from the bolometric magnitude and radius the mass 5 and radiation-pressure $1-\beta = \cdot0675$. (By prearrangement the period was chosen to give an integral mass.) It is only at this stage that the hydrogen content is involved. Knowing now the mass we find ρ_c for the model $n=3$, and hence $\Pi\sqrt{\rho_c}$ (observed), Π being the observed period.

According to the theory of pulsations, $\Pi\sqrt{\rho_c}$ should be very nearly a constant, independent of the star selected. To a closer approximation, however, it is a function of β and increases as the radiation-pressure increases. Since the value of β has been found, we are able in the last line of the table to give the computed value of $\Pi\sqrt{\rho_c}$ for comparison with the observed value.

The new calculation has somewhat increased the difference between the observed and calculated periods. Thus the ratio for δ Cephei, which was 1·7 in my former calculation (with $\Gamma=\frac{5}{3}$), has now become 2·4. I think that the discrepancy is too large to be attributed to any of the ordinary sources of uncertainty, and it does not seem possible to explain it by any reasonable change in the assumed proportion of hydrogen. It seems therefore to be rather strong evidence that the density distribution is less concentrated than in the model $n=3$. As shown by J. C. P. Miller,* a less concentrated model has a longer theoretical period. With the present figures a polytropic distribution with n rather less than 2 would be necessary to give agreement.

9. In these calculations it has been assumed that the absorption-coefficients are given correctly by Kramers's theory, from which the adopted formula (5) for the opacity was obtained. Kramers's results are rather closely confirmed

* *M.N.*, 90, 59, 1929.

by wave-mechanics, but the confirmation is not exact. For wave-mechanics values I take as authoritative the tables given by M. Stobbe.*

For the L absorption of $\overline{Fe}$ or $\overline{Al}$, Stobbe's values are represented by an atomic absorption coefficient $\cdot 0040 Z^4\lambda^3$. Kramers's theory gave $\cdot 0020 Z^4\lambda^3$ (*I.C.S.*, p. 237) ; but faulty weights were used at that time, and with modern weights the coefficient would be ·0030. The error of weight does not affect his calculation of the *emission*, which is the result actually used in our work. It follows that we have been employing an L absorption which is $\frac{3}{4}$ of the wave-mechanics value.

For the K absorption at the wave-lengths which concern us, Stobbe's table gives an atomic coefficient $\cdot 017 - \cdot 019 Z^4\lambda^3$. Kramers's theory gave the coefficient ·0185, or, with modern weights, ·037. Thus the K absorption here employed is double the value according to wave-mechanics. I think it may have been overlooked by those who have regarded Kramers's theory as closely confirmed by wave-mechanics that his numerical results require this correction for faulty weighting.

The corrections to the K and L absorption will to some extent cancel ; but I think that for $\overline{Fe}$ and certainly for $\overline{Al}$ our results depend mainly on the K absorption-coefficient. Accordingly a strict adherence to wave-mechanics would reduce (nearly halve) our adopted opacity. This would raise the proportion of hydrogen in the Sun to 40 per cent., and other stars would be affected similarly.

Degeneracy of the Red Dwarfs

10. Attention may be called to an incidental result reached in the course of this work, viz. that the red dwarfs with masses less than $\frac{1}{4} \times \odot$ are appreciably degenerate (§ 7). The state of ionisation of these stars was investigated by Fowler and Guggenheim,† who concluded that they were highly ionised, *e.g.* iron would retain on the average between 2 and 3 electrons. I think that this result is clearly wrong, and that actually more than half the electrons are retained. It is true that by admitting a large proportion of hydrogen we have decreased the ionisation ; but even without the hydrogen these stars must be much nearer to degeneracy than Fowler and Guggenheim supposed.

Investigations have dealt chiefly with the two extremes, viz. perfectly gaseous matter and highly degenerate matter, and little has been written on the transition state. At present therefore no theory covers the red dwarfs with masses less than a quarter that of the Sun. Presumably as the mass decreases the brightness falls off much more rapidly than if the material continued to be perfect gas.

* *Annalen der Physik*, 5 Folge, 7, 700–701, 1930.
† *M.N.*, 85, 939, 1925.

THE ANALYSIS OF LINE INTENSITIES IN STELLAR SPECTRA.

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In a series of papers in the *Monthly Notices* * and in the Bakerian Lecture,† Professor Milne has given a method of analysing the results of spectro-photometric observations of line contours in stellar spectra. Milne's mathematical work has given precision to the introduction of an electron pressure controlling the ratio of ionised to neutral atoms. Formerly this electron pressure was supposed to represent a mean value in some part of the atmosphere mainly responsible for the formation of the absorption lines in question. Milne takes a particular optical depth, and connects the number of ionised and neutral atoms above this depth with the electron pressure at this particular depth. For the application of these equations to actual stellar spectra we require a method of determining the actual number of atoms in a certain state above a particular optical depth. This is supplied by comparison with a model of the atmosphere—Schuster's model—in which the atoms have no continuous absorption, but exhibit line absorption and are placed above a photosphere which radiates like a black body.

§ 1. Let r_ν be the ratio of the intensity in the stellar spectrum at a frequency ν inside the line to the intensity of the continuous background just outside the line. Then corresponding to every r_ν there is an optical depth τ which is the "optical thickness of the column of the real atmosphere which if placed in front of a photospheric radiating surface and deprived of all opacity, except the power of scattering monochromatically, would produce an intensity ratio r_ν at a frequency ν, equal to that in the actual stellar atmosphere." ‡

It follows that the number of atoms in the Schuster atmosphere (which can be found quickly from the observed value of ν for which r_ν takes a prescribed value, *i.e.* from the width of the line) is actually equal to the number of atoms in the real atmosphere above the optical depth, which is connected with r_ν by the formula

$$\tau = \tfrac{4}{3}\frac{r_\nu}{1 + r_\nu}.$$

The correctness of this formula for τ, however, depends on the vertical distribution of atoms in the stellar atmosphere. It is correct in a special case only, that in which the number of atoms concerned in the formation of the line is proportional to the electron pressure (η constant).

* *M.N.*, 89, 3, 17, 157, 1928.
† *Phil. Trans. Roy. Soc.*, A, 228, 421, 1929. Referred to as M.
‡ M, p. 433.

The number of atoms above a particular optical depth can, however, always be found from the observed value of r_ν without reference to an equivalent Schuster atmosphere. In the present paper it is suggested that a closer approximation to the behaviour of an actual atmosphere can be obtained by replacing methods relying implicitly on the assumption "η constant," by a method which uses the linear approximation,

$$\frac{1}{1+\eta} = \alpha + \beta\tau.$$

§ 2. Let l_ν be the coefficient of line absorption of the actual atmosphere, k the coefficient of continuous absorption.*

Then if a beam of radiation traverses 1 cm. of the atmosphere which has a density ρ, the fraction $(l_\nu + k)\rho$ of the radiation is removed from the beam.

Let

$$\eta = l_\nu / k.$$

Further, let N be the number of atoms of a particular kind of mass M per c.c. of the atmosphere, which has a density ρ.

Let

$$N \,.\, M = \epsilon\rho.$$

Let s_ν be the scattering coefficient of the particular atoms.

Then

$$\eta = \frac{\epsilon s_\nu}{k}.$$

The optical depth τ is defined by

$$d\tau = -k\rho dx.$$

Now let N_0 be the number of atoms of the particular kind contained in a column of unit cross-section, down to the optical depth τ_0.

Then

$$N_0 M = -\int_{\tau=0}^{\tau=\tau_0} \epsilon\rho dx.$$

But

$$d\tau = -k\rho dx\ ;$$

hence

$$-\eta d\tau = \epsilon s_\nu \rho dx,$$

and hence

$$N_0 = \frac{1}{M s_\nu}\int_0^{\tau_0} \eta d\tau. \qquad (2.1)$$

N_0 can therefore be calculated if η is known as a function of τ. If the distribution of the particular atoms in the atmosphere is known, η is known except for a constant factor, which can be determined from the observed

* Eddington, *M.N.*, 89, 620, 1929. Referred to as E.

value of ν for which r_ν takes a prescribed value—that is, from the observed width of the line. For example, suppose

$$\eta = \eta_0(1 + \beta\tau)$$

where β is known. Then from (2.1),

$$N_0 \,.\, Ms_\nu = \eta_0(\tau_0 + \tfrac{1}{2}\beta\tau_0^2),$$

and η_0 is determined from the observed value of r_ν for a given value of ν.

Unfortunately, very few types of vertical distribution lead to an analytical expression for r_ν as a function of η.

If $\mathcal{J}_\nu'(\theta)d\nu$ is the flow of radiation of frequency ν to $\nu + d\nu$ within the line in a direction making an angle θ with the outward direction, $\mathcal{J}(\theta)d\nu$ the corresponding flow outside the line,

$$\mathcal{J}_\nu' = \frac{1}{4\pi}\int \mathcal{J}_\nu'(\theta)d\omega, \qquad H_\nu' = \frac{1}{4\pi}\int \mathcal{J}_\nu'(\theta)\cos\theta\, d\omega.$$

Then * the differential equation governing the behaviour of $\mathcal{J}_\nu'$ is

$$\frac{d}{d\tau}\left(\frac{1}{1+\eta}\frac{d\mathcal{J}_\nu'}{d\tau}\right) = 3(\mathcal{J}_\nu' - \mathcal{J}),$$

with

$$H_\nu' = \frac{1}{3(1+\eta)}\frac{d\mathcal{J}_\nu'}{d\tau},$$

subject to the boundary conditions that $\mathcal{J}_\nu'$ is regular at infinity and $\mathcal{J}_\nu' = 2H_\nu'$ at $\tau = 0$. The required quantity r is equal to $(H_\nu'/H)_{\tau=0}$.

Eddington (*loc. cit.*) has given solutions for the two cases.

(1) η = constant :

$$r_\nu = \frac{1 + \frac{2}{3}q}{1 + \eta + \frac{2}{3}q}. \tag{2.2}$$

Here

$$q^2 = 3(1+\eta)(1+\epsilon\eta),$$

where ϵ is the super-elastic collision constant. Its value is small, and it is important only at the centre of a line. It is neglected in the following discussion, and in the remainder of this paper the symbol ϵ has the meaning given on p. 483.

(2) The case where $1/(1+\eta) = \alpha + \beta\tau$.

Here

$$r_\nu = \frac{a - \sqrt{(\frac{1}{3}\alpha)}(2+\beta)K_0'/K_0}{1 - \sqrt{(\frac{1}{3}\alpha)}2K_0'/K_0}. \tag{2.3}$$

The argument of the Bessel functions $K_0(x)$ and $K_0'(x)(= -K_1'(x))$ is

$$\sqrt{\frac{12\alpha}{\beta^2}}.$$

* E, p. 624.

§ 3. In the case of the uniform atmosphere, η is constant, which is the case if ϵ, the distribution factor, is proportional to $PT^{-\frac{9}{2}}$, so that η, which is equal to $\frac{\epsilon s_\nu}{k}$, is constant. Then from (2.1),

$$N_0 = \frac{1}{Ms_\nu}\int_0^{\tau_0} \eta d\tau = \frac{\eta\tau_0}{Ms_\nu}. \tag{3.1}$$

Milne's formula for the introduction of an equivalent Schuster atmosphere gives the correct result in this case. A Schuster atmosphere is one in which there is no continuous absorption, or $k=0$. The atmosphere is a cloud of atoms possessing line absorption but no continuous absorption, placed in front of a surface radiating like a black body. If there are N^* atoms over each sq. cm. of the radiating surface, then in the system of approximations used by Milne, *

$$1/r_\nu = 1 + \tfrac{3}{4}N^*Ms_\nu. \tag{3.2}$$

Using Milne's approximations, we have instead of Eddington's result (2.2) the equation

$$r_\nu = \sqrt{\left(\frac{1}{1+\eta}\right)} \text{ (approximately).} \tag{3.3}$$

Then from (3.1)

$$N_0 = \frac{\tau_0}{Ms_\nu}\left\{\frac{1}{r_\nu^2} - 1\right\}. \tag{3.4}$$

Now $N = N^*$ if we choose τ_0 such that

$$\tau_0\left\{\frac{1}{r_\nu^2} - 1\right\} = \frac{4}{3}\left\{\frac{1}{r_\nu} - 1\right\}$$

or

$$\tau_0 = \frac{4}{3}\frac{r_\nu}{1+r_\nu}. \tag{3.5}$$

The value of τ_0 given by (3.5) is the value given by Milne.†

This procedure is dependent on the assumption η = constant.

Let

$$1/(1+\eta) = \alpha + \beta\tau.$$

Then

$$\int_0^{\tau_0} \eta d\tau = \frac{1}{\beta}\log\left(1 + \frac{\beta\tau_0}{\alpha}\right) - \tau_0,$$

$$N_0 = \frac{1}{Ms_\nu}\left\{\frac{1}{\beta}\log\left(1 + \frac{\beta\tau_0}{\alpha}\right) - \tau_0\right\}. \tag{3.6}$$

If β/α is known from the ionisation and excitation formulæ of the particular atoms concerned, both α and β can be found from the observed value of ν for which r_ν takes a prescribed value, from equation (3.3),

$$r_\nu = \frac{\alpha - \sqrt{(\frac{1}{3}\alpha)}(2+\beta)K_0'/K_0}{1 - \sqrt{(\frac{1}{3}\alpha)}2K_0'/K_0}.$$

* M, p. 433. † M, equation (4.7).

Numerical examples show that Milne's optical depth,

$$\tau_0 = \tfrac{4}{3}\frac{r_\nu}{1 + r_\nu},$$

does not give $N = N^*$.† Direct calculation of N from (3.3) and (3.6) is preferable to the introduction of an intermediate N^*.

§ 4. *Maximum of the Balmer Lines*

Milne ‡ has given the precise definition of the maximum of an absorption line. Fix the attention on the point Q' of the line contour where the intensity ratio r has some assigned value r'. Let ν' be the frequency at this point, and ν_0 the frequency of the line centre, Q_0. As the line increases in intensity in a series of stellar atmospheres, Q' separates more and more from Q_0, and then approaches Q_0 again. The temperature at which Q' reaches its maximum separation from Q_0 is the temperature of the absorption line for this value r' of r.

It was therefore predicted by Milne that the temperature at which the Balmer lines appear as a maximum would depend on whether the wings of the lines or the centres were under consideration. Miss Payne and Miss Williams have found the maximum to shift from type A0 for $r = 0{\cdot}48$ to types A3 – A5 for $r = 0{\cdot}96$.

Consider a sequence of atmospheres with increasing temperature, such that η is constant in each atmosphere. As we go along the sequence, the number of atoms of a particular kind increases, comes to a maximum, and decreases. Since $\eta = \dfrac{NMl_\nu}{\rho}$, for a fixed value of $\nu - \nu_0$, η increases, comes to a maximum, and decreases. Let T be the temperature at which the line has a maximum width at $r = r_1$, corresponding to $\eta = \eta_1$. Let this maximum width be $\nu_1 - \nu_0$ in frequency units. Consider any other ordinate $r = r_2$, corresponding to $\eta = \eta_2$; let the width be $\nu_2 - \nu_0$. Now consider further any other temperature T'; let the widths of the line produced by the atmosphere at this temperature be $\nu_1' - \nu_0$ and $\nu_2' - \nu_0$ respectively.

Since the line has a maximum width at $r = r_1$ at the temperature T,

$$\nu_1' - \nu_0 < \nu_1 - \nu_0, \quad \text{hence} \quad l_{\nu_1'} > l_{\nu_1},$$

$l(\nu - \nu_0)$ being a decreasing function of $\nu - \nu_0$.

But

$$\eta_1 = \frac{NMl_{\nu_1}}{\rho} = \frac{N'Ml_{\nu_1'}}{\rho'}$$

and

$$\eta_2 = \frac{NMl_{\nu_2}}{\rho} = \frac{N'Ml_{\nu_2'}}{\rho'};$$

† Values of N and N^* are shown for a variety of cases in Tables II, III and VI.

‡ *M.N.*, **89**, 33, 1928.

hence, given $l_{\nu_1}' > l_{\nu_1}$,

$$\frac{N}{\rho} > \frac{N'}{\rho'};$$

hence

$$l_{\nu_2}' > l_{\nu_2},$$

hence

$$\nu_2' - \nu_0 < \nu_2 - \nu_0$$

—*i.e.* the line has a maximum width at $r = r_2$ at the same temperature T.

Thus, in a series of *uniform* atmospheres with $\eta =$ constant in each, the maxima must occur at the same temperature for all values of r. The observed shift of maximum as we go from centre to wing is therefore an observation of an effect dependent on a departure from uniformity. We cannot legitimately examine the problem by making use of Milne's optical depth

$$\tau_0 = \tfrac{4}{3}\frac{r_\nu}{1 + r_\nu}$$

which is only valid for uniform atmospheres, with $\eta =$ constant. The variation of temperature with optical depth is given by a well-known equation,*

$$T^4 = T_0^4(1 + \tfrac{3}{2}\tau).$$

Thus T is known as a function of τ. Further, the electron pressure P is a known function of the optical depth τ. Assuming that the atmosphere consists mainly of singly ionised atoms, $P = \frac{1}{2}p$, where p is the total gas pressure.

Now

$$dp = -g\rho dx,$$

hence

$$kdp = gd\tau.$$

Putting

$$k = \frac{\kappa P}{(RT)^{\frac{3}{2}}} \quad \text{and} \quad P = \tfrac{1}{2}p,$$

$$P^2 = \frac{g}{\kappa}(RT)^{\frac{3}{2}}(\tau + \tfrac{3}{4}\tau^2)$$

nearly. Here R is Boltzmann's constant.

We can now calculate x_0 and x_1, the fractions of neutral and singly ionised hydrogen atoms. Let χ_1 be the ionisation potential of hydrogen $= 13{\cdot}85$ volts. Then let

$$K_1 = \frac{q_1}{q_0}\frac{(2\pi m)^{\frac{3}{2}}}{h^3}(RT)^{\frac{3}{2}} \exp -\left(\frac{\chi_1}{RT}\right),$$

where q_1, q_0 are the statistical weights of the ionised and neutral states, m the mass of the electron, and h is Planck's constant.

We then have

$$\frac{x_1}{x_0}P = K_1,$$

so that we can calculate x_1 and x_0 as functions of τ.

* Eddington, *Internal Constitution of the Stars*, p. 323.

Of the neutral atoms, a certain fraction $N_0^{(s)}/N_0$ will be in the excited state, whose excitation potential is $E^{(s)}$. If $q_0^{(s)}$ is the weight of this state,

$$\frac{N_0^{(s)}}{N_0} = \frac{q_0^{(s)}}{q_0} \exp - \left\{\frac{E^{(s)}}{RT}\right\}.$$

The number of atoms in the state $E^{(s)}$ is therefore known as a function of τ. We suppose that the proportion of the total number of hydrogen atoms, in all states of ionisation and excitation, to those of other elements, is constant throughout the atmosphere.

Taking a definite case, suppose $T = 10{,}000°$ at $\tau = \frac{1}{3}$. Calculating T at $\tau = \frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$, we find $T = 9300°$, $9550°$ and $10{,}400°$ at these points. The calculation of the number of hydrogen atoms in the Balmer state $E^{(s)} = 10{\cdot}155$ volts is shown in Table I*a*, the last number, $x_0 N_0^{(s)}$, containing an arbitrary factor.

TABLE I*a*

Hydrogen Atoms in Balmer State, $E^{(s)} = 10{\cdot}155$ volts

τ	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
T	9300°	9550°	10000°	10400°
P	1050	1500	2100	2600
K_1	140	220	500	1000
x_0	0·88	0·87	0·81	0·72
x_1	0·12	0·13	0·19	0·28
$x_0 N_0^{(s)}$	27	34	62	87

Extending the calculations to different values of T at $\tau = \frac{1}{3}$, viz. 11,000°, 12,000° and 13,000°, we get Table I*b* which shows the concentration of hydrogen atoms in the Balmer state from level to level with change of temperature.

TABLE I*b*

Concentration of Hydrogen Atoms in Balmer State, $E^{(s)} = 10{\cdot}155$ volts

τ	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{2}$
T at $\tau = \frac{1}{3}$.				
10,000	20	27	62	87
11,000	49	61	111	125
12,000	67	82	117	120
13,000	57	70	46	50

The maximum concentration at any particular optical depth is not sharp. From Table I*b* we find that at $\tau = \frac{1}{24}$ the maximum concentration occurs when $T = 12{,}400$ at $\tau = \frac{1}{3}$: the maximum concentration at $\tau = \frac{1}{2}$ occurs in a slightly cooler atmosphere—that is, one in which $T = 11{,}500°$ at $\tau = \frac{1}{3}$.

Finally

$$\eta = \frac{N_0^{(s)} M s_\nu}{k\rho};$$

we suppose

$$\rho \propto (N_0 + N_1),$$

so that

$$\frac{N_0^{(s)}}{\rho} \propto x_0 \exp - \left\{\frac{E^{(s)}}{RT}\right\}.$$

We choose s_ν to make $\eta = 1$ at $\tau = \frac{1}{3}$, and then trace the variation of η with τ. This is shown in Table I*c*.

TABLE I*c*

Variation of η with Optical Depth

τ T at $\tau=\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
10,000	0·63	0·72	1·0	1·39
11,000	0·73	0·83	1·0	1·16
12,000	1·17	1·06	1·0	0·99

To construct the contour of the line, given a particular atmosphere, we have to choose α and β to give as good a fit as possible to the equation

$$\frac{1}{1+\eta} = \alpha + \beta\tau.$$

α and β have to be chosen afresh for every value given to η at the optical depth $\tau = \frac{1}{3}$, that is to say, for every point on the line. The accuracy with which the linear formula $\alpha + \beta\tau$ fits the calculated values of $\frac{1}{1+\eta}$ varies, but is, on the whole, extremely good.

TABLE II

Detailed Calculations for $T = 10{,}000$ at $\tau = \frac{1}{3}$

Value of η at $\tau=\frac{1}{3}$	$\frac{1}{1+\eta}$	Comparison of Table I with $\alpha+\beta\tau$ $\tau=\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	r_ν	$N_0 M s_\nu$	$N^* M s_\nu$
$\frac{1}{10}$	·954 − ·142τ	·941 *·942*	·933 *·931*	·909 *·909*	·878 *·883*	0·94	0·06	0·08
$\frac{1}{3}$	·823 − ·320τ	·826 *·826*	·806 *·800*	·750 *·746*	·685 *·693*	0·83	0·24	0·27
1	·650 − ·460τ	·613 *·612*	·581 *·573*	·500 *·497*	·418 *·430*	0·71	0·54	0·54
3	·377 − ·370τ	·346 *·346*	·316 *·315*	·250 *·254*	·193 *·192*	0·57	1·31	1·01

Note to Table II.—The figures in plain type are values of $\frac{1}{1+\eta}$ from direct calculation. Those in italics are from the formula $\alpha + \beta\tau$, as given in column 2.

Detailed calculations for $T = 10{,}000$ at $\tau = \frac{1}{3}$ are given in Table II. To exhibit the change of temperature for maximum width with the change from centre to wing it is necessary to calculate values of r_ν for the remaining temperatures at the extreme points, *i.e.* taking $\eta = \frac{1}{10}$ and 3 respectively at the optical depth $\tau = \frac{1}{3}$. These are tabulated in Table III, and contours drawn

TABLE III

Change of Contour with Temperature

T	η at $\tau=\frac{1}{3}$	$\frac{1}{1+\eta}$	r_ν	$N_0 Ms_\nu$	$N^* Ms_\nu$
10,000	$\frac{1}{10}$	$\cdot954-\cdot142\tau$	0·94	0·06	0·085
	3	$\cdot377-\cdot370\tau$	0·57	1·31	1·01
11,000	$\frac{1}{10}$	$\cdot939-\cdot088\tau$	0·95	0·07	0·07
	3	$\cdot325-\cdot220\tau$	0·56	1·30	1·05
12,000	$\frac{1}{10}$	$\cdot892+\cdot060\tau$	0·965	0·07	0·05
	3	$\cdot215+\cdot110\tau$	0·51	1·41	1·28

for $T=10{,}000$ and for $T=12{,}000$ are shown in fig. 1. From Table III it is seen that with increasing temperature and a given value of η at $\tau=\frac{1}{3}$, the lines become shallower in the wings but deeper at the centre. On account of the differences in abundance of atoms at the depth $\tau=\frac{1}{3}$, as shown in

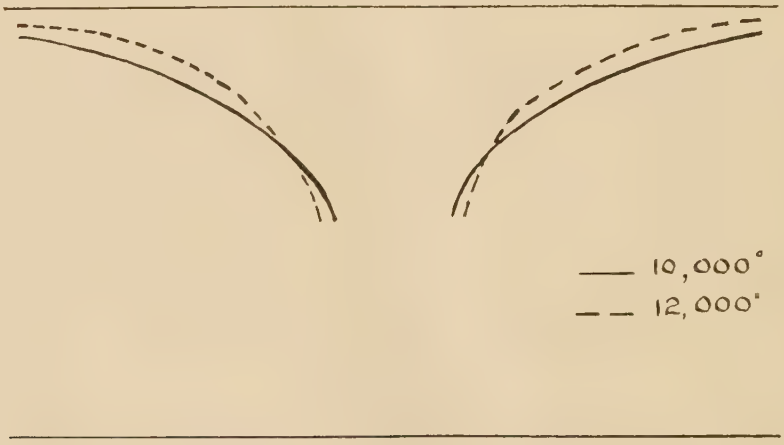

FIG. 1.—*Balmer Lines at* 10,000° *and* 12,000°.

Table I*a*, the frequency at which $\eta=1$ at $\tau=\frac{1}{3}$ changes from temperature to temperature. We can take the distance from the centre of the line, in frequency, to be proportional to the square root of the number of atoms. Using the contour of the line at 10,000° as a guide we can convert, approximately, the differences in r_ν in Table III into differences in width of the line at a constant r_ν—*e.g.* at $r=0{\cdot}96$ the difference $\Delta r=0{\cdot}025$ corresponds to the frequency ratio

$$\frac{\nu''-\nu_0}{\nu'-\nu_0}=\frac{4}{3}.$$

This ratio can also be found by giving η at $\tau=\frac{1}{3}$ the value $0{\cdot}1\times(\frac{4}{3})^2$ and using Table I. Then $\frac{1}{1+\eta}=\cdot84+\cdot03\tau$, which gives $r=\cdot94$. The relative

widths of the lines at $T = 10{,}000°$ and $T = 12{,}000°$ are therefore in the ratio

$$\frac{(\nu-\nu_0)_{10,000}}{(\nu-\nu_0)_{12,000}} = \frac{4}{3}\sqrt{\frac{117}{62}},$$

the relative numbers of atoms, 117 and 62, being found in Table I*a*.

We arrive in this way at the relative widths of the lines, for different temperatures, at constant r_ν. We find that at $r = 0{\cdot}5$ the maximum width occurs at about 11,700°; and at $r = 0{\cdot}96$ at about 10,900°.

Comparing these figures with Miss Payne's results we find that in Ao stars, $T = 11{,}700°$ at $\tau = \frac{1}{3}$; and in types A3 – A5, $T = 10{,}900°$ at $\tau = \frac{1}{3}$.

These temperatures depend on the assumed value of g/k'. But the detailed working out establishes the change of maximum with change of temperature in a quantitative way, and illustrates the value of the approximation, $\frac{1}{1+\eta} = \alpha + \beta\tau$.

Numbers of Atoms above a Particular Optical Depth.—The number of atoms above the optical depth $\tau_0 = \frac{4}{3}\frac{r_\nu}{1+r_\nu}$ can be calculated from the formula (3.6)—

$$N_0 M s_\nu = \frac{1}{\beta}\log\left(1+\frac{\beta\tau_0}{\alpha}\right) - \tau_0.$$

The results are shown in Tables II and III; the values of $N^* M s_\nu$ are also tabulated, where

$$N^* M s_\nu = \frac{4}{3}\left\{\frac{1}{r_\nu} - 1\right\}.$$

The differences between N and N^* are not large, but the use of the form $\frac{1}{1+\eta} = \alpha + \beta\tau$ is a second approximation to the actual atmosphere; the use of N^*, which is equivalent to the form "η constant," being the usual approximation.

It is worth remarking that there is one point on the contour where $N^* = N$, and that this point moves towards the wing with increasing temperature.

§ 5. Milne's method of analysing stellar spectra may be summarised briefly as follows. Formulæ are given for N_0, the number of atoms above the optical depth τ_0, for different conditions of the stellar atmosphere. N_0 is to be determined from the observed width of the line from the pair of equations :

$$N_0 M s_\nu = \frac{4}{3}\left(\frac{1}{r} - 1\right), \tag{5.11}$$

$$\tau_0 = \frac{4}{3}\frac{r}{1+r}. \tag{5.12}$$

We have seen that (5.12) holds only for a special case; the pair of equations could, however, be replaced by the more accurate pair,

$$N_0 M s_\nu = \frac{1}{\beta}\log_e\left(1+\frac{\beta\tau_0}{\alpha}\right) - \tau_0 \tag{5.13}$$

and (2.3).

Milne's formulæ for N_0, however, take no account of change of temperature with optical depth. Since the practical part of the contour—say $\cdot 9 > r > \cdot 5$—is formed almost entirely in the region $\tau > \frac{1}{100}$ (a statement which will shortly be justified), it is formed in a region where the variation in K is as great as the variation in P. Increased accuracy is therefore to be gained by taking account of the variation of K with τ.

Consider the following method :—

Let $\eta = \eta_0 f(\tau)$ where $f(\tau) = 1$ when $\tau = \frac{1}{3}$. Suppose that we are able to solve the differential equation

$$\frac{d}{d\tau}\left(\frac{1}{1+\eta}\frac{d\mathcal{J}_\nu'}{d\tau}\right) = 3(\mathcal{J}_\nu' - \mathcal{J}), \tag{5.2}$$

subject to the appropriate boundary conditions, whatever the form of $f(\tau)$, the solution being

$$r = \phi(\eta_0). \tag{5.3}$$

Select a particular element and calculate its distribution throughout an atmosphere, in both neutral and ionised states, assuming trial values for T at $\tau = \frac{1}{3}$ and for g/k'. Putting $\eta_0 = 1$, let the result for the neutral atom be

$$\left.\begin{aligned}\eta_1 &= f_1(\tau)\\ \eta_2 &= f_2(\tau)\end{aligned}\right\}. \tag{5.4}$$

and for the ionised atom,

Then let the solutions of (5.2) be

$$r_1 = \phi_1(1), \qquad r_2 = \phi_2(1).$$

Let $\Delta\nu_1$ be the observed width of the neutral line at $r = r_1$.

Let $\Delta\nu_2$ be the observed width of the ionised line at $r = r_2$.

Now r_1 and r_2 have been chosen to make $\eta_1 = \eta_2 = 1$ at $\tau = \frac{1}{3}$. But

$$\eta_1 = \frac{\epsilon_1 s(\Delta\nu_1)}{k} = \frac{N_1 M}{\rho}\frac{s(\Delta\nu_1)}{k},$$

$$\eta_2 = \frac{\epsilon_2 s(\Delta\nu_2)}{k} = \frac{N_2 M}{\rho}\frac{s(\Delta\nu_2)}{k}.$$

Hence, at $\tau = \frac{1}{3}$

$$\frac{N_1}{N_2} = \frac{s(\Delta\nu_1)}{s(\Delta\nu_2)} = \frac{(\Delta\nu_2)^2}{(\Delta\nu_1)^2},$$

adopting the form

$$s(\Delta\nu) = \frac{s_0}{a^2\Delta\nu^2}.$$

On account of the variation of temperature and electron pressure with optical depth, the ratio of the numbers of atoms in the neutral and ionised states can receive interpretation only if the optical depth at which this ratio occurs is known. Further, on account of differences in vertical distribution of neutral and ionised atoms, measurement of the ratio of the widths of the neutral and ionised lines at the *same* argument r does not give directly the

ratio N_1/N_2 at any prescribed optical depth (though the particular optical depth at which $\frac{N_1}{N_2}=\frac{\Delta\nu_2{}^2}{\Delta\nu_1{}^2}$ could be calculated for any given r). But corresponding to any prescribed optical depth τ_0 there is a family of pairs of points Q, Q' such that the ratio of the widths of the lines taken at any pair of points is equal to the square root of the ratio of the number of neutral to ionised atoms per c.c. at the depth τ_0, one pair of points Q, Q' corresponding to every value of η at the optical depth τ_0.

That is to say, we can make the widths of the ionised and neutral lines give the relative numbers of ionised and neutral atoms at any optical depth we please, if only we know at what points to measure the lines.

This method of analysing stellar spectra contains only the assumptions hitherto employed in all methods, that the numbers of atoms in the neutral and ionised states can be calculated from the ionisation formulæ which are valid for thermodynamical equilibrium, and that the atmosphere is well mixed. It avoids the assumption "η constant," and takes account of the variation of temperature with optical depth. The only drawback is the fact that at the present time the solutions of (5.2) can be expressed in a convenient form only in two cases: η constant and $\frac{1}{1+\eta}=\alpha+\beta\tau$. This drawback is, however, implicitly contained in all present methods of analysing spectra, except Professor Pannekoek's numerical solution of the equations.

The remainder of this paper is devoted to an examination of the extent to which the formula

$$\frac{1}{1+\eta}=\alpha+\beta\tau$$

can be made satisfactorily to represent calculated distributions of neutral and ionised atoms, and to the calculation of the values of r_1 and r_2.

§ 6. To perform a test of the part played by the atmosphere above $\tau=\frac{1}{100}$ on the one hand, and by the part below $\tau=\frac{1}{2}$ on the other, we construct an atmosphere consisting of two parts. Above a boundary at $\tau=\tau_1$, η has the value η_1, and below the boundary, $\eta=\eta_2$.

Let

$$q_1{}^2=3(1+\eta_1) \quad \text{and} \quad q_2{}^2=3(1+\eta_2),$$
$$q_2(1+\eta_1)+q_1(1+\eta_2)=Q,$$
$$q_2(1+\eta_1)-q_1(1+\eta_2)=R.$$

Then the value of r is given by the formula *

$$r=\frac{1}{1+\eta_1}\left\{1+q_1\frac{\frac{2}{3}\eta_1\{R+Q\exp 2q_1\tau_1\}-2(\eta_2-\eta_1)(1+\eta_1)}{(1+\eta_1+\tfrac{2}{3}q_1)Q\exp.\ 2q_1\tau_1-(1+\eta_1-\tfrac{2}{3}q_1)R}\right\}. \qquad (6.1)$$

For the upper layer, set $\tau_1=\frac{1}{100}$ in (6.1).

Let $\eta_2=1$, $\eta_1=5$; (6.1) gives $r=0{\cdot}71$.

* See *M.N.*, **90**, 182–183, 1929, for the derivation of these formulæ (6.1 and 6.2). The standard solution of the homogeneous atmosphere is given in E (*M.N.*, 89, 620, 1929).

For a simple atmosphere with η (constant) $=1$, $r=0{\cdot}72$; and with $\eta=5$, $r=0{\cdot}43$. The upper layer has small effect.

Alternatively,* we can investigate the effect of the upper layer by calculating N, the number of atoms above $\tau=\frac{1}{100}$ in the actual atmosphere, and then calculating r for a composite atmosphere, in the lower layer of which η is constant and in the upper $k=0$ but there are N atoms at a low pressure (a Schuster layer). Let $\sigma_1=NMs_\nu$.

$$r_\nu=\frac{1+\frac{2}{3}q}{1+\eta+\frac{2}{3}q(1+\frac{3}{2})\sigma_1}. \tag{6.2}$$

As an example, for Fe^+ in the Sun, σ_1 is of order 10^{-2}. The upper layer contributes less than 1 per cent. to the value of r.

For the lower layer, set $\tau_1=\frac{1}{2}$.

With

$$\eta_2=2, \quad \eta_1=1, \quad r=0{\cdot}72;$$
$$\eta_2=5, \quad \eta_1=1, \quad r=0{\cdot}70.$$

Since $r=0{\cdot}72$ when $\eta=1$ throughout the atmosphere, the change caused by doubling the value of η below $\tau=\frac{1}{2}$ is extremely small. Even increasing its value fivefold has only a slight effect.

These examples make it clear that a formula representing the behaviour of η between $\tau=\frac{1}{2}$ and $\tau=\frac{1}{100}$ will give a reliable value of r when $r=0{\cdot}7$.

For larger values of r the relevant layer of the atmosphere is shifted to greater optical depth, and the value at $\tau=\frac{1}{2}$ becomes important.

In some cases (at the temperatures here considered in the case of the ionised atom) it is difficult to draw a straight line through the calculated points when $\frac{1}{1+\eta}$ is plotted against τ. It is, however, easy to divide the atmosphere into two sections and to represent the calculated points by a straight line in each section. An example is shown in fig. 2.

Consider an atmosphere in which $\frac{1}{1+\eta}=\alpha+\beta\tau$ when $\tau>\tau_1$, and $\frac{1}{1+\eta}=\lambda+\mu\tau$ when $\tau<\tau_1$. Then, below $\tau=\tau_1$,†

$$\mathcal{J}_\nu'=H(2+\beta+3\tau)+AK_0(x_1),$$
$$H_\nu'=H(\alpha+\beta\tau)+A\{\tfrac{1}{3}(\alpha+\beta\tau)\}^{\frac{1}{2}}K_0'(x_1),$$

where

$$x_1=\left\{\frac{12(\alpha+\beta\tau)}{\beta^2}\right\}^{\frac{1}{2}}.$$

Above $\tau=\tau_1$,

$$\mathcal{J}_\nu'=H(2+\mu+3\tau)+CK_0(x_2)+DI_0(x_2),$$
$$H_\nu'=H(\lambda+\mu\tau)+\{\tfrac{1}{3}(\lambda+\mu\tau)\}^{\frac{1}{2}}\{CK_0'(x_2)+DI_0'(x_2)\},$$

where

$$x_2=\left\{\frac{12(\lambda+\mu\tau)}{\mu^2}\right\}^{\frac{1}{2}}.$$

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* See note, p. 493.

† The standard solution for $\frac{1}{1+\eta}=\alpha+\beta\tau$ is given in E, p. 630.

The boundary conditions are the continuity of $\mathcal{J}_\nu'$ and of H_ν' at $\tau=\tau_1$, and the condition $\mathcal{J}_\nu'=2H_\nu'$ at $\tau=0$. These give

$$H(\mu-\beta)=AK_0(x_1')-CK_0(x_2')-DI_0(x_2'), \tag{6.31}$$

$$0=AK_1(x_1')-CK_1(x_2')+DI_1(x_2'), \tag{6.32}$$

$$\begin{aligned}-H(2+\mu-2\lambda)=&C\{K_0(x_0')+2\sqrt{(\tfrac{1}{3}\lambda)}\,.\,K_1(x_0')\}\\&+D\{I_0(x_0')-2\sqrt{(\tfrac{1}{3}\lambda)}I_1(x_0')\},\end{aligned} \tag{6.33}$$

where

$$x_1'=\left[\frac{12(\alpha+\beta\tau_1)}{\beta^2}\right]^{\frac{1}{2}},\qquad x_2'=\left[\frac{12(\lambda+\mu\tau_1)}{\mu^2}\right]^{\frac{1}{2}},\qquad x_0'=\left[\frac{12\lambda}{\mu^2}\right]^{\frac{1}{2}}.$$

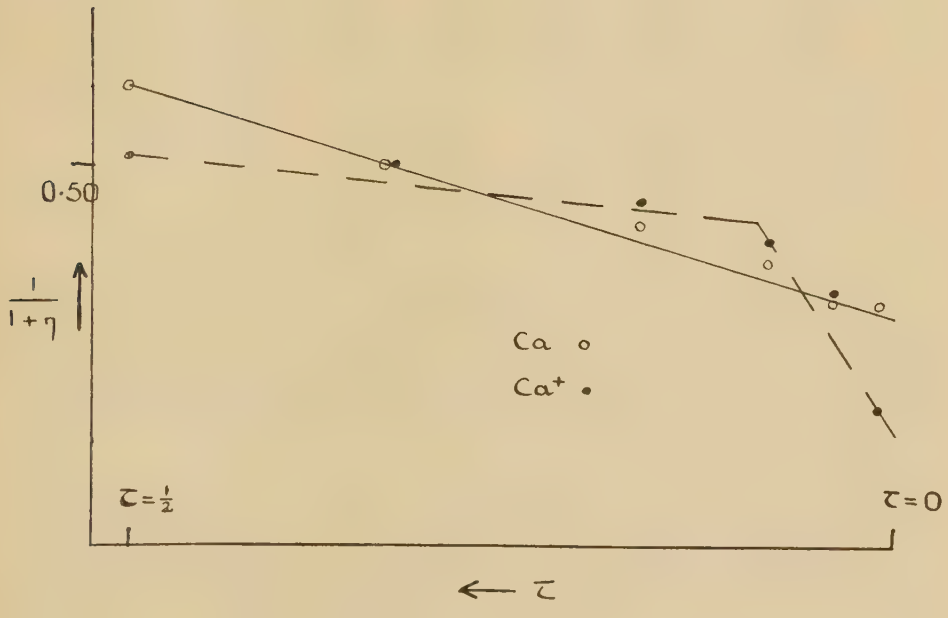

FIG. 2.—*Distribution of* Ca *and* Ca^+ *in the Sun.*

The value of r is found by eliminating A and determining C and D in terms of H. Then

$$r=\lambda+\sqrt{(\tfrac{1}{3}\lambda)}\{-\frac{C}{H}K_1(x_0)+\frac{D}{H}I_1(x_0)\}. \tag{6.4}$$

As an example, take the case illustrated in fig. 2, which applies to Ca^+ in the Sun (6000°).

$$\begin{aligned}\frac{1}{1+\eta}&=0{\cdot}42+0{\cdot}18\tau \quad \text{below} \quad \tau=0{\cdot}095\,;\\&=0{\cdot}15+3{\cdot}0\tau \quad \text{above} \quad \tau=0{\cdot}095.\end{aligned}$$

$$r=0{\cdot}67.$$

In a simple atmosphere in which $\frac{1}{1+\eta}=0{\cdot}42+0{\cdot}18\tau$ throughout, $r=0{\cdot}71$.

§ 7. *Metals in Late-type Stars.*—Table IV shows details of the calculations for *Ca* and Ca^+ in the Sun. $T=6000$ at $\tau=\frac{1}{3}$, $g=2{\cdot}74\,.\,10^4$ and $\kappa'=0{\cdot}017$.

TABLE IV

Ca and Ca^+ in Go-type Stars (the Sun)

τ	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{100}$
T	6240	6000	5730	5570	5460	5460
K	13100	7600	4300	3000	2300	2300
P	1050	820	550	380	260	127
K/P	12·45	8·7	7·8	8·9	8·7	18·1
$Ca\ x_0$	0·074	0·103	0·114	0·112	0·103	0·052
$Ca^+\ x_1$	0·926	0·897	0·886	0·888	0·897	0·948
η, Ca	0·67	1	1·36	1·69	2·07	2·10
η, Ca^+	0·96	1	1·20	1·53	2·06	4·4
$1/1+\eta\ Ca$	0·60	0·50	0·42	0·37	0·32	0·32
$1/1+\eta\ Ca^+$	0·51	0·50	0·45	0·40	0·33	0·18

The formulæ used in the calculations are set out in § 4. The values of $1/1+\eta$ for *Ca* and for Ca^+ are plotted against τ in fig. 2. It is seen that in the case of the neutral atom, the approximation $1/1+\eta=\alpha+\beta\tau$ is extremely good; for the ionised atom, the atmosphere must be broken up into two parts, as shown in the figure. Calcium has the ionisation potential 6·08 volts.

Calculations similar to those shown in Table IV have been made for titanium (6·8 volts) and strontium (5·65 volts). In the case of titanium, the distribution of atoms in excited states, with excitation potentials 1, 2 and 3 volts, were examined; and in the case of strontium, the temperature at $\tau=\frac{1}{3}$ was taken successively as 5000°, 6000° and 7000°, and a possible absolute magnitude effect * was investigated by making a set of values with $g=2{\cdot}74\,.\,10^2$.

Whatever the conditions in the case of the neutral atom, if η has the value 1 at $\tau=\frac{1}{3}$, then the value of r is 0·72, to within about 1 per cent. Calculated values are shown in Table V.

The distribution of the ionised atoms is awkward, as can be seen from fig. 2. It cannot be conveniently represented by one straight line, and in each case the atmosphere has been broken up into two parts and the formulæ (6.3) used. The results, however, are as strikingly simple as in the case of the neutral atoms; whatever the conditions, if η has the value 1 at $\tau=\frac{1}{3}$, then the value of r is ·66 within 1 per cent. Tabulated values are given in Table V.

* Not, of course, the ordinary absolute magnitude effect. We are here looking for a possible absolute magnitude effect on the position at which to measure the width of the line.

TABLE V

Metals in G-type Stars

Values of r *when* $\eta = 1$ *at* $\tau = \frac{1}{3}$

Element	Excitation Potential	T at $\tau = \frac{1}{3}$	$\alpha + \beta\tau$	r
Ti	0	6000	$\cdot30 + 0\cdot55\tau$	$\cdot71_0$
	1 volt	,,	$\cdot33 + 0\cdot48\tau$	$\cdot71_5$
	2 volts	,,	$\cdot41 + 0\cdot24\tau$	$\cdot72_0$
	3 ,,	,,	$\cdot43 + 0\cdot20\tau$	$\cdot72_5$
Ca	0	6000	$\cdot30 + 0\cdot55\tau$	$\cdot71$
Sr	0	7000	$\cdot35 + 0\cdot44\tau$	$\cdot72_2$
		6000	$\cdot32 + 0\cdot52\tau$	$\cdot71_8$
		5000	$\cdot32 + 0\cdot52\tau$	$\cdot71_8$
Sr *		5000	$\cdot29 + 0\cdot62\tau$	$\cdot72_1$
Ti^+	0	6000	$\cdot45 + 0\cdot12\tau$ $\cdot14 + 3\cdot2\tau$	$\cdot67$
	3 volts	,,	$\cdot58 - 0\cdot28\tau$ $\cdot24 + 4\cdot4\tau$	$\cdot67$
Ca^+	0	6000	$\cdot42 + 0\cdot18\tau$ $\cdot15 + 3\cdot0\tau$	$\cdot65$
Sr^+	0	7000	$\cdot41 + 0\cdot22\tau$ $\cdot18 + 2\cdot6\tau$	$\cdot67$
		6000	$\cdot41 + 0\cdot23\tau$ $\cdot17 + 2\cdot7\tau$	$\cdot67$
		5000	$\cdot46 + 0\cdot08\tau$ $\cdot14 + 3\cdot28\tau$	$\cdot66$
Sr^+ *		5000	$\cdot41 + 0\cdot22\tau$ $\cdot16 + 2\cdot8\tau$	$\cdot66$

* $g = 2\cdot74 \times 10^2$.

The relative numbers of atoms in the ionised and neutral states at the particular optical depth $\tau = \frac{1}{3}$ are determined by measuring the width of the line due to the ionised atom at $r = \cdot67$ and the width of the line due to the neutral atom at $r = \cdot72$.

Assuming Unsöld's law, that the coefficient of scattering is of the form

$$s_\nu = \frac{s_0}{a^2(\nu_1 - \nu_0)^2},$$

let N_1 be the number of neutral atoms per c.c. at $\tau = \frac{1}{3}$, N_2 be the number of ionised atoms per c.c. at $\tau = \frac{1}{3}$, $\Delta\nu_2$ be the width of the ionised line, measured at $r = \cdot67$, $\Delta\nu_1$ be the width of the neutral line, measured at $r = \cdot72$.

Then, since

$$\eta = \frac{NM}{k\rho} s(\Delta\nu)$$

and since the measurements have been conducted in such a way as to make $\eta_1 = \eta_2 = 1$ at $\tau = \frac{1}{3}$, we have

$$\frac{N_2}{N_1} = \frac{\Delta\nu_1^2}{\Delta\nu_2^2}.$$

As was pointed out in § 5, a knowledge of the ratios for a pair of elements determines T at $\tau = \frac{1}{3}$ and g/κ'. In practice, as many elements as possible should be used to get the best values of T at $\tau = \frac{1}{3}$ and g/κ'. It is clear, from the insensitiveness of the values of r in Table V to the initial temperature chosen, that no second approximation is necessary.

Further pairs of points can be obtained by giving η different values at $\tau = \frac{1}{3}$. Taking $\eta = \frac{1}{3}$ at $\tau = \frac{1}{3}$ and taking $T = 6000°$ at $\tau = \frac{1}{3}$ and proceeding as before, we find for

$$\begin{aligned} Ca, &\quad r_1 = 0{\cdot}90_5; \\ Ca^+, &\quad r_2 = 0{\cdot}86_5; \\ Sr, &\quad r_1 = 0{\cdot}90; \\ Sr^+, &\quad r_2 = 0{\cdot}86_5. \end{aligned}$$

Hence, the line due to the ionised atom, measured at $r_1 = {\cdot}86$, and the line due to the neutral atom, measured at $r_2 = {\cdot}90$, give again the conditions at $\tau = \frac{1}{3}$. Similar sets of points can be calculated for other values of τ.

To compare these results with those obtained using Milne's procedure, N and N^*, as defined in § 3, have been calculated from Table V. The calculation is slightly complicated, in the case of the ionised atom, by the double atmosphere. The results are shown in Table VI. T at $\tau = \frac{1}{3}$ is taken as 6000° throughout, $g = 2{\cdot}74 \cdot 10^4$, and the excitation potential is zero.

TABLE VI

*Values of N and N**

Ti	$N = {\cdot}69$	$N^* = {\cdot}55$	Ti^+	$N = {\cdot}77$	$N^* = {\cdot}64$
Ca	·71	·55	Ca^+	·74	·64
Sr	·68	·53	Sr^+	·74	·69

The unit in which both N and N^* are expressed is $(Ms_\nu)^{-1}$.

The significant difference is that between $\frac{N_1^*}{N_1}$ and $\frac{N_2^*}{N_2}$, where N_1, N_2 refer to the neutral and ionised atom respectively. These quantities differ by 4 per cent. in the case of titanium and 24 per cent. in the case of strontium.

§ 8. *A Colour Correction.*—The expressions (2.3), (2.2) for r involve a certain approximation, namely, that $\mathcal{J} = \mathcal{J}_T$; that is to say, the radiation just outside the line has equilibrium density. This assumption is correct in only one region of the spectrum (6400–6900 A. in the Sun). The point is that discussed by Eddington.*

If we put $\mathcal{J}_T = a + b\tau$ the differential equation becomes

$$\frac{d}{d\tau}\left(\frac{1}{1+\eta}\frac{d\mathcal{J}_\nu'}{d\tau}\right) = 3\{\mathcal{J}_\nu' - (a+b\tau)\},$$

* *M.N.*, 89, 623, 1929.

where

$$\frac{1}{1+\eta} = \alpha + \beta\tau.$$

The solution, subject to the usual boundary conditions, is

$$\mathcal{J}_\nu' = a + \tfrac{1}{3}b\beta + b\tau + AK_0(x),$$

and r is given by

$$r = \frac{1}{H}\,\frac{\tfrac{1}{3}b\alpha - \sqrt{(\tfrac{1}{3}\alpha)}(\alpha + \tfrac{1}{3}b\beta)K_0'/K_0}{1 - \sqrt{(\tfrac{1}{3}\alpha)}2K_0'/K_0}; \tag{8.1}$$

the argument x_0 of the Bessel functions $K_0(x_0)$ and $K_0'(x_0)$ is $\sqrt{(12\alpha/\beta^2)}$.

Further,

$$H = \frac{b + a\sqrt{3}}{3 + 2\sqrt{3}} \quad \text{and} \quad \frac{b}{a} = \frac{3}{8}\frac{h\nu}{kT}. \tag{8.2}$$

Similarly, equations (6.3), (6.4) require modification to the following set :—

$$\frac{b}{3}(\mu - \beta) = AK_0(x_1') - CK_0(x_2') - DI_0(x_2'), \tag{8.31}$$

$$0 = AK_1(x_1') - CK_1(x_2') + DI_1(x_2'), \tag{8.32}$$

$$-a - \frac{b}{3}(\mu - 2\lambda) = C\{K_0(x_0') + 2\sqrt{(\tfrac{1}{3}\lambda)}K_1(x_0')\} + D\{I_0(x_0') - 2\sqrt{(\tfrac{1}{3}\lambda)}I_1(x_0')\}, \tag{8·33}$$

x_0', x_1' and x_2' having the same values as before. r is given by

$$r = \frac{b}{3}\lambda + \sqrt{(\tfrac{1}{3}\lambda)}\left\{-\frac{C}{H}K_1(x_0) + \frac{D}{H}I_1(x_0)\right\}, \tag{8.4}$$

with $T = 6000$, $\lambda = 4000$ A., $\frac{a}{H} = 1{\cdot}63$ and $\frac{b}{H} = 3{\cdot}66$. The following values of r, worked for the standard case, 6500 A., and for 4000 A., show the correction :—

TABLE VII

Colour Correction

	6500 A.	4000 A.
Ca	·71	$\cdot 69_5$
Ca$^+$	·65	·63
Sr	$\cdot 71_8$	$\cdot 70_8$
Sr$^+$	·67	$\cdot 65_5$

Since the colour correction is small, always in the same sense, and is approximately always of the same amount, it may be neglected. It may be regarded as changing the value of η at $\tau = \frac{1}{3}$ by the same amount for both neutral and ionised lines, leaving the value of N_2/N_1 unaltered ; unless, of course, we are comparing lines in different parts of the spectrum, such as a neutral line in the red, and an ionised line in the violet, in which case the correction should be applied to the value of r at which the line in the violet is to be measured.

Summary

Eddington's model, with $\frac{I}{I+\eta} = \alpha + \beta\tau$, is applied to stellar atmospheres, α and β being chosen to fit calculated distributions of neutral and ionised atoms. The maximum of the Balmer lines in A-type stars is accounted for quantitatively. General considerations show that, corresponding to any prescribed optical depth, there exist pairs of points r_1 and r_2 at which to measure the widths of the lines, such that the square of the ratio of the widths, measured at these points, gives the ratio of the number of neutral to ionised atoms at this particular optical depth. Hence, from pairs of lines from different elements, T and g can be found for the star. Calculations are shown for neutral and ionised strontium, calcium and titanium in G-type stars. In all cases examined, measurements of the neutral line at $r_1 = \cdot 71$ and of the ionised line at $r = \cdot 66$ give the ratio of neutral to ionised atoms at $\tau = \frac{1}{3}$.

CONDENSATIONS IN AN EXPANDING UNIVERSE.

G. C. McVittie, Ph.D.

Introduction

The following paper consists of two parts. In the first section we develop the theory of expanding or contracting universes which contain a non-uniform spherically symmetric distribution of mass about a point and in which the pressure is isotropic. The line-elements thus found are classified into external and internal solutions. By "internal" solutions we mean those which correspond to regions near the centre of the condensation where the equilibrium depends essentially on the density of the condensation. "External" solutions apply to regions distant from the centre where the mass of the whole condensation is the essential factor in the equilibrium. In all cases there is a flow of the matter towards or away from the centre of the condensation according as the universe is expanding or contracting.

When the condensation is sufficiently small compared with the size of the universe as a whole, we show that the radius of the condensation is independent of the radius of the universe. It also follows, in this case, that condensations are of two types, viz. one type which has a similar condensation around the antipodes of its centre, and another with a "hole" around the antipodes. By a "hole" is meant a region where the density is less than in the surrounding more distant parts of the universe. This latter type of condensation is not dealt with in detail, as we wish to retain the property of universes with uniform distribution of matter, namely, that both halves of the universe are similar.

In the second section of the paper we attempt to apply the "external" line-element for condensations to those actually found in physical space. By calculating the velocity of mass-motion at the surface of the Sun or of a

spiral nebula, in a universe expanding at the rate at which ours appears to be expanding at present, we show that the assumption of isotropy of pressure could not hold. Hence the solutions found cannot apply to the present state of the universe. We then calculate that, for a mass equal to that of an average spiral nebula, the maximum radius is 2×10^{23} cm., and the maximum possible fractional rate of expansion is about one-thousandth part of what it is to-day. These values are arrived at on the assumption that the pressure in the universe as a whole is still approximately isotropic. Finally, we calculate that the time-lapse from the formation of the nebula, with radius as above, to the time when the universe first attains a condition similar to its present one, is of the order of 10^{10} years. These results appear to lead to a rather smaller original size and a shorter time-scale for the nebulæ than those obtained by Jeans.

The conclusion is that the line-elements found can only apply to the earliest stages of the formation of condensations in an Einstein universe which has started to expand, and that the process of formation of the actual nebulæ was probably complete before the radius of the universe had expanded to 1·5 times its original value.

I. The Theory of Condensations

(1) *The Form of the Line-element*

In this section we seek the general form of line-element which could correspond to a non-uniform distribution of matter in an expanding or contracting universe, the distribution being spherically symmetric about a point, which we take as the origin of the "polar" co-ordinates we employ.

In the first place, we know that spherically symmetric distributions of matter whose Newtonian gravitational potential is Ω have line-elements of the form (in the first approximation)

$$ds^2 = (1 - 2\Omega)dt^2 - \frac{1}{c^2}(1 + 2\Omega)\{dy^2 + y^2(d\theta^2 + \sin^2\theta d\phi^2)\}, \qquad (1)$$

where y is a polar co-ordinate measured from the centre of the distribution. In (1) we neglect the curvature and expansion of space.

We can reduce the form (1) in the following way. Put

$$q = \{1 + \Omega(y)\}y$$

and neglect squares of Ω and $\frac{d\Omega}{dy}$. Then (1) becomes

$$ds^2 = \{1 - 2\Omega(q)\}dt^2 - \frac{1}{c^2}\left\{\frac{dq^2}{1 + 2q\Omega'(q)} + q^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}, \qquad (2)$$

where a dash denotes differentiation with respect to q.

In the second place, we have that the line-element of an expanding or contracting universe, filled with a uniform distribution of matter, is

$$ds^2 = dt^2 - \frac{a^2(t)R^2}{c^2}\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}. \qquad (3)$$

In this, R is the radius of space at the moment when expansion or contraction began. This is supposed to occur when $a(t) = 1$ and, subsequently, $a(t)R$ is the actual radius at time t. c is the velocity of light, assumed constant throughout the history of the universe, and $a(t)R\chi$ measures spatial distance. The universe is supposed to extend from $\chi = 0$ to $\chi = \pi$.

We can deduce two other forms of the line-element (3). If we put

$$z = 2R \tan \frac{\chi}{2}, \tag{4}$$

we get the symmetrical form

$$ds^2 = dt^2 - \frac{a^2(t)}{c^2}\left\{\frac{dz^2 + z^2(d\theta^2 + \sin^2\theta d\phi^2)}{(1 + z^2/4R^2)^2}\right\}; \tag{5}$$

while by means of

$$r = \frac{z}{1 + z^2/4R^2} = R \sin \chi \tag{6}$$

we obtain

$$ds^2 = dt^2 - \frac{a^2(t)}{c^2}\left\{\frac{dr^2}{1 - r^2/R^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}. \tag{7}$$

We note that the range $0 \leqslant \chi \leqslant \pi$ corresponds to the range $0 \leqslant z \leqslant \infty$, but that the equivalent range of r is zero to R and back to zero again. So that, whereas the co-ordinates χ and z have one value for each point in space, the co-ordinate r has one value for both the given point and its antipodean point. This property of not distinguishing between a point and its antipodes leads us to employ the co-ordinate r and the line-element (7) in what follows.

Inspection of the forms (2) and (7) suggests the following form of line-elements for the metric of an expanding or contracting universe with a non-uniform spherically symmetric distribution of matter, viz. :

$$ds^2 = \left\{1 - \frac{\kappa}{c}V(r)\right\}dt^2 - \frac{a^2(t)}{c^2}\left\{\frac{dr^2}{\mu(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}, \tag{8}$$

where $V(r)$ is given by an equation

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -\sigma(r), \tag{9}$$

and is therefore the Newtonian potential of a distribution of density $\sigma(r)$. The function $\mu(r)$ is restricted in two ways. First, $\mu(r)$ must reduce to $(1 - r^2/R^2)$ when $V(r) = 0$; and secondly, $\mu(r)$ must reduce to $\left\{1 + \frac{\kappa}{c}qV'(q)\right\}$ when the curvature of space is neglected and $q = a(t)r$. In the first case the metric (8) will reduce to (7), whilst in the second it will become (2).

The units for measuring $V(r)$ and $\frac{\kappa}{c}$ in (8) and (9) are such that, if $\sigma(r)$ is expressed in ergs/cm.^{3}, then

$$\frac{\kappa}{c} = \frac{8\pi G}{c^4} = 2 \times 10^{-48} \text{ c.g.s. units}, \tag{10}$$

where G is the gravitation constant.

We shall refer to $\sigma(r)$ as the "auxiliary density," defining the condensation in the expanding universe.

(2) *The Isotropy of Pressure*

The energy-tensor $T_\mu{}^\nu$ for the matter in the universe is given by Einstein's equations

$$-\kappa c T_\mu{}^\nu = G_\mu{}^\nu - \tfrac{1}{2}(G-2\lambda)\delta_\mu{}^\nu, \tag{11}$$

where $G_\mu{}^\nu$ is the mixed Riemann-Christoffel tensor for the metric of the universe. Now the distributions of matter with either of the line-elements (2) or (7) possess the property that the pressure is isotropic. We shall therefore take as our fundamental assumption, regarding the state of the matter in the universe whose line-element is (8), that the pressure in the universe is isotropic. This can be expressed by the condition

$$T_1{}^1 = T_2{}^2 = T_3{}^3, \tag{12}$$

where the several $T_\mu{}^\nu$ are calculated by (11) for the metric (8). We shall see that the two equations (12) reduce to a single one for the metric (8), and this is regarded as the equation for determining $\mu(r)$ when $\sigma(r)$ is given.

It remains to note that, when $\Omega(q) = \dfrac{m}{q}$ in (2), we are dealing with an empty region outside a condensation of mass m. We speak of this as the "external" solution for the condensation. All other solutions, in which $\Omega(q)$ is a solution of Poisson's equation (9), are regarded as "internal" solutions for the region inside the boundary of the condensation. In the same way we expect to obtain internal and external solutions for the condensation in universes with metric (8).

(3) *Solution of the Equation of Isotropy of Pressure*

For both external and internal solutions we have the following non-zero components of the energy-tensor for the metric (8) :—

$$\kappa c \rho_{00} = +\kappa c T_4{}^4 = +\frac{3\dot{a}^2}{a^2}\frac{1}{\zeta} + \frac{c^2}{a^2}\left\{\frac{1}{r^2} - \frac{\mu'}{r} - \frac{\mu}{r^2}\right\} - \lambda, \tag{13}$$

$$\kappa c p_1 = -\kappa c T_1{}^1 = -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\frac{1}{\zeta} - \frac{c^2}{a^2}\left\{\frac{1}{r^2} - \frac{\mu}{r^2} - \frac{\zeta'}{r\zeta}\mu\right\} + \lambda, \tag{14}$$

$$\begin{aligned}\kappa c p_2 = -\kappa c T_2{}^2 = -\kappa c T_3{}^3 = &-\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}_2}{a^2}\right)\frac{1}{\zeta} \\ &- \frac{c^2}{a^2}\left\{\tfrac{1}{4}\left(\frac{\zeta'}{\zeta}\right)^2\mu - \tfrac{1}{2}\frac{\zeta'}{\zeta r}\mu - \tfrac{1}{2}\frac{\mu'}{r} - \tfrac{1}{4}\frac{\zeta'}{\zeta}\mu' - \tfrac{1}{2}\frac{\zeta''}{\zeta}\mu\right\} + \lambda,\end{aligned} \tag{15}$$

$$-\kappa c T_4{}^1 = +c^2\frac{\dot{a}}{a^3}\frac{\zeta'}{\zeta}\mu, \qquad -\kappa c T_1{}^4 = -\frac{\dot{a}}{a}\frac{\zeta'}{\zeta^2}. \tag{16}$$

where $\zeta = 1 - \dfrac{\kappa}{c}V(r)$, and a dot denotes differentiation with respect to t,

whilst a dash denotes differentiation with respect to r. Here ρ_{00} is the density, and p_1, p_2 the pressures.

The equations (12) then reduce to the single equation

$$\frac{1}{r^2}-\frac{\mu}{r^2}-\frac{\zeta'\mu}{r\zeta}=\tfrac{1}{4}\left(\frac{\zeta'}{\zeta}\right)^2\mu-\tfrac{1}{2}\frac{\zeta'\mu}{r\zeta}-\tfrac{1}{2}\frac{\mu'}{r}-\tfrac{1}{4}\frac{\zeta'\mu'}{\zeta}-\tfrac{1}{2}\frac{\zeta''}{\zeta}\mu \tag{17}$$

by (14) and (15). This can be written as

$$\frac{d\mu}{dr}-\frac{2\mu\left\{\tfrac{1}{4}\left(\frac{\zeta'}{\zeta}\right)^2-\tfrac{1}{2}\frac{\zeta'}{r\zeta}-\tfrac{1}{2}\frac{\zeta''}{\zeta}-\frac{1}{r^2}\right\}}{\frac{1}{r}+\tfrac{1}{2}\frac{\zeta'}{\zeta}}=-\frac{2}{r^2\left(\frac{1}{r}+\tfrac{1}{2}\frac{\zeta'}{\zeta}\right)}. \tag{18}$$

(a) *The Internal Solution.*—For an internal solution we make the usual assumptions for the equilibrium of a fluid gravitating sphere,* viz. that the density-function $\sigma(r)$, and the functions V, V' derived from $\sigma(r)$ by means of equation (9), can have no singularities within the boundary of the condensation. We are here, of course, only dealing with the case when $\sigma(r)$ remains the same function of r throughout the condensation. Under these conditions (9) gives

$$V'(r)=-\frac{\int_0^r\sigma(r)r^2dr}{r^2}=-\frac{M(r)}{r^2} \tag{19}$$

and

$$V(r)=-a^2-\int_0^r\frac{M(r)}{r^2}dr, \tag{20}$$

where a^2 is a constant of integration and $M(r)$ is the mass, expressed in ergs, of the auxiliary density distribution inside radius r.

Since

$$\zeta=1-\frac{\kappa}{c}V(r),$$

we have

$$\zeta'=-\frac{\kappa}{c}V'$$

and

$$\zeta''=-\frac{2\zeta'}{r}+\frac{\kappa\sigma(r)}{c}. \tag{21}$$

Substituting these expressions into (18) and rearranging the terms in the coefficient of μ, we obtain

$$\frac{d\mu}{dr}-2\mu\left\{\frac{1}{r}+\tfrac{1}{2}\frac{\zeta'}{\zeta}+\tfrac{1}{2}\frac{\left(\frac{\zeta'}{r\zeta}-\frac{\kappa\sigma}{c\zeta}\right)}{\frac{1}{r}+\tfrac{1}{2}\frac{\zeta'}{\zeta}}\right\}=-\frac{2}{r^2\left(\frac{1}{r}+\tfrac{1}{2}\frac{\zeta'}{\zeta}\right)}.$$

* A. S. Eddington, *Internal Constitution of the Stars*, chap. iv.

The integrating factor of this equation is

$$\frac{1}{r^2\zeta}exp\left[-\int\left(\frac{\zeta'-\frac{\kappa\sigma(r)r}{c}}{\zeta+\frac{1}{2}r\zeta'}\right)dr\right].$$

Now, in virtue of (21) we have

$$\frac{d}{dr}(\zeta+\tfrac{1}{2}r\zeta')=\tfrac{1}{2}\left(\zeta'+\frac{\kappa\sigma(r)r}{c}\right).$$

Making use of this result, the integrating factor becomes

$$\frac{exp\left[2\frac{\kappa}{c}\int\frac{\sigma(r)rdr}{\zeta+\frac{1}{2}r\zeta'}\right]}{r^2\zeta(\zeta+\frac{1}{2}r\zeta')^2}.$$

Hence the solution of (18) is

$$\mu=r^2\zeta(\zeta+\tfrac{1}{2}r\zeta')^2\,exp\left[-\frac{2\kappa}{c}\int\frac{\sigma(r)rdr}{\zeta+\frac{1}{2}r\zeta'}\right]\left\{-\frac{1}{R^2}-\int\left(\frac{2\,exp\left[2\frac{\kappa}{c}\int\frac{\sigma(r)rdr}{\zeta+\frac{1}{2}r\zeta'}\right]}{r^3(\zeta+\frac{1}{2}r\zeta')^3}\right)dr\right\},$$

where R^2 is a positive constant of integration. We put

$$f(r)=\frac{exp\left[2\frac{\kappa}{c}\int\frac{\sigma(r)rdr}{\zeta+\frac{1}{2}r\zeta'}\right]}{(\zeta+\frac{1}{2}r\zeta')^3}. \tag{22}$$

By integrating the last term in the expression for μ once by parts, we obtain

$$\mu=\frac{\zeta}{\zeta+\frac{1}{2}r\zeta'}\left\{1-\frac{r^2}{f(r)}\left(\frac{1}{R^2}+\int\frac{f'(r)dr}{r^2}\right)\right\}. \tag{23}$$

Hence the metric for the internal solution is

$$\left.\begin{aligned} ds^2&=\zeta dt^2-\frac{a^2(t)}{c^2}\left\{\frac{dr^2}{\mu(r)}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right\}\\ \text{where}\quad \zeta&=1-\frac{\kappa}{c}V(r)\\ V&=-a^2-\int_0^r\frac{M(r)dr}{r^2}\quad\text{and}\quad M(r)=\int_0^r\sigma(r)r^2dr \end{aligned}\right\}, \tag{24}$$

whilst $\mu(r)$, $f(r)$ are given by (23) and (22) respectively.

We can interpret the constant R as the initial radius of the universe. For if we neglect all terms in (24) which involve $\frac{\kappa}{c}\sigma(r)$ and put $a^2=0$, we get $f(r)=1$ and $f'(r)=0$, so that $\zeta=1$ and $\mu=1-r^2/R^2$. The metric of (24) reduces to (7). We shall see later (equation (36)) that if we neglect the terms in $\frac{1}{R^2}$ in (24) we can reduce the metric to the form (2) in the first approximation.

It is to be noticed that when the condensation is present the zero value of μ does not occur for $r=R$ but, by (23), for the value of r which makes

$$1-\frac{r^2}{f(r)}\left(\frac{1}{R^2}+\int\frac{f'(r)}{r^2}dr\right)=0. \tag{25}$$

Since the magnitude of $\sigma(r)$ is, as yet, arbitrary, we cannot say that any of the solutions of (25) will bear any relation to the value $r=R$. Hence, if we wish to regard (24) as the solution corresponding to a universe with a condensation, whose undisturbed condition is given by (7), we must suppose that $\sigma(r)$ is small enough to give a root of order R in (25). We can then look upon the functions $f(r)$ and $\int\frac{f'(r)}{r^2}dr$ in (25) as giving rise to perturbation-terms of order $\frac{\kappa}{c}$, $\left(\frac{\kappa}{c}\right)^2$, etc. We shall suppose henceforward that $\sigma(r)$ is of this nature.

We remark also that the actual density inside the condensation is not $\sigma(r)$, but ρ_{00}, given by (13) with the values (23), (24) of μ, ζ. The auxiliary density merely enables us to distinguish between one condensation and another. The pressure is, similarly, given by (14) or (15), since now

$$p=p_1=p_2.$$

It is of importance to observe that in all cases T_4^1, T_1^4 given by (16) are not zero unless $\dot{a}=0$. This means that the matter in the universe has a velocity as a whole, towards or away from the centre of the condensation. This result is of interest in the physical application of metrics of type (24).

(b) *Two Types of Condensation.*—We consider the case when (24) can be looked upon as being the metric of an approximately spherical universe of radius R_1, where R_1 is the root of order R of equation (25). Then the values of the density and pressure given by (13), (14) will differ from the undisturbed values corresponding to metric (7), only in small terms of order $\frac{\kappa}{c}$, $\left(\frac{\kappa}{c}\right)^2$, etc. Under these conditions it is obvious that we are not *a priori* restricted to positive values of $\sigma(r)$ alone. For all that is necessary is that ρ_{00} and p should be positive, and this could be achieved by a suitable choice of $a(t)$ whether $\sigma(r)$ were positive or not. We shall call a condensation in which $\sigma(r)$ is negative a " hole."

Consider now the antipodes of the origin in the (approximately spherical) universe whose metric is (24). Since the co-ordinate r represents both the origin and its antipodes by the single value $r=0$, it follows that, on separating out the origin and its antipodes by the use of some co-ordinate such as z (equation (6)), there would be a condensation at both points. Let us put, therefore,

$$r=\frac{z}{1+z^2/4R_1^2}=\frac{\bar{z}}{1+\bar{z}^2/4R_1^2}, \tag{26}$$

where

$$z\bar{z}=4R_1^2.$$

Then the origin is $z=0$ or $\bar{z}=\infty$, and its antipodes is $z=\infty$ or $\bar{z}=0$. Also it follows from (26) that $\sigma(r)$ transforms into the same function, Σ, of z or of $\bar{z}$.

Let, now, $r=r_0$ be some point, P, inside the condensation. Then, since the universe is still approximately spherical, $r=r_0$ corresponds to $z=z_1$ or z_2, and to $\bar{z}=\bar{z}_1$ or $\bar{z}_2$, say, so that z_1, $\bar{z}_2$ refer to the point P, and z_2, $\bar{z}_1$ refer to its antipodes, P'. Also by (26), if z_1, $\bar{z}_1$ are each less than $2R_1$, z_2, $\bar{z}_2$ must be greater, and *vice versa*. Since P, P' are antipodean points, we must also have $z_1=\bar{z}_1$ and $z_2=\bar{z}_2$.

We now have two possibilities for the behaviour of the function Σ as z goes through the value $2R_1$ (i.e. $r=R_1$):

(A) $\Sigma(z)$ does not change sign. Then, since

$$\Sigma(z_2)=\Sigma(\bar{z}_1)=\Sigma(z_1),$$

the auxiliary density has the same value at P' as it has at P. Thus the condensations around the origin and its antipodes are similar.

(B) $\Sigma(z)$ changes sign. Suppose $\Sigma(z_1)$ is positive, then $\Sigma(z_2)$ must be negative, since $z_1<2R_1<z_2$. But $\Sigma(z_2)=\Sigma(\bar{z}_1)$. Hence $\Sigma(z_1)=-\Sigma(\bar{z}_1)$. Thus a condensation at the origin has a " hole " at its antipodes and, of course, the opposite could also hold.

We conclude that a condensation at the origin must have either a similar condensation or a " hole " at its antipodes, and the existence of one or the other depends on the behaviour of the auxiliary density-function at $r=R_1$.

(c) *The External Solution.*—We shall give this name to any solution of (18) obtained by assuming that V is of the form

$$V=\frac{m}{r}h\left(\frac{r}{R}\right), \tag{27}$$

so that

$$-\sigma_e(r)=V''+\frac{2}{r}V'(r)=\frac{m}{rR^2}h''\left(\frac{r}{R}\right). \tag{28}$$

It is assumed that $h\left(\frac{r}{R}\right)\to 1$ as $R\to\infty$, and that $h''\left(\frac{r}{R}\right)$ remains finite and non-zero as $R\to\infty$.

In the solutions of this type all the terms in the expressions for the density and pressure which give the deviation from the values corresponding to (7) are proportional to powers m. Further, it can be shown (§ 4) that m is equal to the auxiliary mass inside the "boundary" of a condensation whose internal solution is given, whatever the law of density within the condensation may be. These are just the properties possessed by the Schwarzschild solution * for the (empty) space surrounding a gravitating body of mass m ergs. It is for these reasons that we call solutions of (18), obtained by assuming (27), "external" solutions, and look upon them as giving the line-element in those parts of the universe where the total auxiliary mass, and not the auxiliary density, of the condensation is the determining factor in the equilibrium.

* A. S. Eddington, *The Mathematical Theory of Relativity*, § 38, 2nd ed., 1924.

Since there are two types of condensation (the types (A) and (B) of section (b)), we expect, *a priori*, two types of external solution also. We find that, if we put

$$h\left(\frac{r}{R}\right)=\text{constant},$$

we obtain * a rigorous solution of (18) in the form

$$ds^2=\left(1-\frac{\kappa m}{cr}\right)dt^2-\frac{a^2(t)}{c^2}\left\{\frac{dr^2}{\left(1-\frac{\kappa m}{cr}\right)\left(1-\left(r-\frac{\kappa m}{2c}\right)^2\Big/R^2\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right\}. \quad (29)$$

It can be shown † that this solution corresponds to two similar condensations of auxiliary mass m, one at the origin and one at its antipodes. Also the external auxiliary density is equal to zero, by (28). Hence (29) appears to be the simplest possible form of external solution for condensations of type (A).

A rigorous external solution for condensations of type (B) cannot be obtained in so simple a form. But if we neglect powers of $\kappa m/c$ higher than the first, we obtain ‡ the following approximate solution in terms of the co-ordinate z given by (6) :—

$$ds^2=\left\{1-\frac{\kappa m}{c}\left(\frac{1}{z}-\frac{z}{4R^2}\right)\right\}dt^2-\frac{a^2(t)}{c^2}\left(\frac{1+\frac{\kappa m}{c}\left(\frac{1}{z}-\frac{z}{4R^2}\right)}{(1+z^2/4R^2)^2}\right)\{dz^2+z^2(d\theta^2+\sin^2\theta d\phi^2)\}. \quad (30)$$

If desired, the rigorous solution could be obtained as follows. We have, converting from z to r,

$$V=m\left(\frac{1}{z}-\frac{z}{4R^2}\right)=\frac{m}{r}\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}; \quad (31)$$

then by (28) we get the external auxiliary density

$$\sigma_e(r)=\frac{m}{rR^2(1-r^2/R^2)^{\frac{3}{2}}}=\frac{m(1+z^2/4R^2)^4}{R^2z(1-z^2/4R^2)^3}. \quad (32)$$

With this expression for $\sigma_e(r)$ we can evaluate $f(r)$ for the problem by (22), and hence, finally, $\mu(r)$ by (23); but the expressions are excessively complicated.

It is obvious that $\sigma_e(r)$ changes sign as z goes through the value $2R$, so that (30) is appropriate for the external solution to a condensation of type (B). That this is so can also be shown by actual transformation of the metric.§

If we allow R to become infinite and suppose $a(t)$ to vary slowly, then, at any instant, the metric (29) reduces rigorously, and (30) reduces approximately, to the ordinary Schwarzschild solution for the field around a body of mass m.

* W. H. McCrea and G. C. McVittie, *M.N.*, **91**, 128–133, 1930.
† W. H. McCrea and G. C. McVittie, *ibid.*, **92**, 7–12, 1931.
‡ G. C. McVittie, *ibid.*, **91**, 274–283, 1931.
§ W. H. McCrea and G. C. McVittie, *loc. cit.*

Henceforward we shall only consider condensations of type (A) with external solution (29). This type of condensation is not only simpler from the mathematical point of view, but possesses the property of preserving the similarity of both halves of the universe. Its external solution also has the property of the Schwarzschild solution, viz. that the external auxiliary density $\sigma_e(r)$ is zero. With a view to physical applications it therefore appears as though this type of condensation would give the simpler results.

(4) *The Equations of Fit*

It now remains to show that the metric (29) can be fitted on to any internal solution of type (A) given by (24), still looking on both these metrics as perturbed forms of (5).

The equations of fit, as is known, must express the fact that the density and pressure are continuous at the boundary of the regions whose metrics are, respectively, (29) and (24). If we denote by ζ_e, μ_e the values of ζ, μ for (29), and ζ_i, μ_i the corresponding functions for (24), the conditions we require are

$$\zeta_e = \zeta_i, \qquad \mu_e = \mu_i, \qquad \zeta_e' = \zeta_i', \qquad \mu_e' = \mu_i'. \tag{33}$$

The equations (33) follow from the equations (13) and (14). We assume, of course, that the function $a(t)$ is the same for both metrics.

The four equations (33) are to be regarded as giving the four unknowns, viz. (1) the radius of the condensation, which must be less than R; (2) the value of m; (3) the ratio of the co-ordinate "r" employed in (29) to that employed in (24); (4) the value of the constant a^2 which occurs in (24).

We do not attempt the general solution of (33), but restrict ourselves to the case when all terms involving powers of $\left(\frac{\kappa}{c}\right)$ higher than the first are neglected. If this is done, it is possible to obtain a solution of (33) without knowing the form of $\sigma(r)$, the internal auxiliary density.

Let us denote by u the variable r employed in (29), so that the metric becomes

$$ds^2 = \left(1 - \frac{\kappa m}{cu}\right)dt^2 - \frac{a^2}{c^2}\left\{\frac{du^2}{\left(1 - \frac{\kappa m}{cu}\right)\left(1 - \left(u - \frac{\kappa m}{2c}\right)^2 \Big/ R^2\right)} + u^2(d\theta^2 + \sin^2\theta d\phi^2)\right\}. \tag{29a}$$

Here, therefore, we have

$$\zeta_e = 1 - \frac{\kappa m}{cu} \quad \text{and} \quad \mu_e = 1 - \frac{u^2}{R^2} - \frac{\kappa m}{c}\left(\frac{1}{u} - \frac{u}{R^2}\right), \tag{34}$$

where powers of κ/c higher than the first are neglected.

Let $\sigma(r)$ be the auxiliary density for the internal solution (24). Then

$$\zeta_i = 1 - \frac{\kappa}{c}V = 1 + \frac{\kappa}{c}\left(a^2 + \int_0^r \frac{M(r)}{r^2}dr\right). \tag{35}$$

We calculate μ_i as follows. From (35) we have

$$\frac{1}{\zeta+\frac{1}{2}r\zeta'}=1+\frac{\kappa}{c}\left(V+\frac{rV'}{2}\right).$$

Hence, using (22), we obtain

$$f(r)=1+\frac{\kappa}{c}\left\{2\int\sigma(r)rdr+3\left(V+\frac{rV'}{2}\right)\right\}.$$

Introducing these into (23), we have

$$\mu_i=\left(1+\frac{\kappa}{c}\frac{rV'}{2}\right)\left\{1-\frac{r^2}{R^2}+\frac{\kappa}{c}\frac{r^2}{2}\int\left(\frac{V''}{r}-\frac{V'}{r^2}\right)dr+\frac{\kappa}{c}\frac{r^2}{R^2}\left(2\int\sigma(r)rdr+3\left(V+\frac{rV'}{2}\right)\right)\right\}$$

$$=1-\frac{r^2}{R^2}+\frac{\kappa}{c}rV'+\frac{\kappa}{c}r^2\frac{\left(2\int\sigma(r)rdr+3V+rV'\right)}{R^2}.$$

Again, using (9), we obtain

$$\begin{aligned}\int\sigma(r)rdr&=-\int(V''r+2V')dr\\&=-2V-rV'+\int V'dr\\&=-V-rV'.\end{aligned}$$

Hence, finally,

$$\mu_i=1-\frac{r^2}{R^2}+\frac{\kappa}{c}rV'+\frac{\kappa}{c}r^2\frac{(V-rV')}{R^2}. \tag{36}$$

Let $r=r_0$ be the boundary of the condensation, and denote the corresponding value of u by u_0. Let V_0, V_0', V_0'' be the values of V, V', V'' at $r=r_0$. Then the equations (33) become

$$1-\frac{\kappa}{c}V_0=1-\frac{\kappa}{c}\frac{m}{u_0}. \tag{37}$$

$$-\frac{\kappa}{c}V_0'=\frac{\kappa m}{cu_0^2}. \tag{38}$$

$$1-\frac{r_0^2}{R^2}+\frac{\kappa}{c}\left\{V_0'r_0+\frac{r_0^2(V_0-r_0V_0')}{R^2}\right\}=1-\frac{u_0^2}{R^2}-\frac{\kappa m}{c}\left(\frac{1}{u_0}-\frac{u_0}{R^2}\right). \tag{39}$$

$$-\frac{2r_0}{R^2}+\frac{\kappa}{c}\left\{r_0V_0''+V_0'+\frac{2r_0}{R^2}(V_0-r_0V_0')-\frac{r_0^3V_0''}{R^2}\right\}=-\frac{2u_0}{R^2}+\frac{\kappa}{c}m\left(\frac{1}{u_0^2}+\frac{1}{R_2}\right). \tag{40}$$

Let $r_0=ku_0$, where k is a constant to be determined. Then (38) gives, in virtue of (19),

$$m=-\frac{r_0^2V_0'}{k^2}=\frac{M(r_0)}{k^2}. \tag{41}$$

Equations (37) and (41) give

$$k=-\frac{r_0V_0'}{V_0}. \tag{42}$$

Also (39) becomes, using (37) and (42),

$$\left(1-\frac{1}{k^2}\right)\frac{r_0^2}{R^2}=\frac{\kappa}{c}\left\{1+\frac{r_0^2}{R^2}\left(1-\frac{1}{k^2}\right)-k\left(1-\frac{r_0^2}{R^2}\right)\right\}V_0. \tag{43}$$

The equation (40) can be reduced, by use of (9), (38) and (42), to

$$\frac{2r_0}{R^2}\left(\frac{1}{k}-1\right)=\frac{\kappa}{c}\left\{r_0\sigma(r_0)\left(1-\frac{r_0^2}{R^2}\right)-\frac{1}{k^2}\frac{V_0 r_0}{R^2}\right\}. \tag{44}$$

From (43), (44) it is obvious that $\left(\frac{1}{k}-1\right)$ is a constant of order κ/c. Put

$$k=1+b\frac{\kappa}{c}, \tag{45}$$

where b is a constant of order unity.

Substituting (45) into (43) we obtain immediately

$$b=\frac{V_0}{2}, \tag{46}$$

whilst the equation (44) becomes

$$-2b\frac{r_0}{R^2}=-\frac{V_0 r_0}{R^2}+r_0\sigma(r_0)\left(1-\frac{r_0^2}{R^2}\right). \tag{47}$$

By the use of (46) we see that, provided $r_0 \neq R$ and $r_0 \neq 0$,

$$\sigma(r_0)=0. \tag{48}$$

In the formulæ (41) and (42) we can put $k=1$, and we obtain

$$m=M(r_0) \tag{41a}$$

and

$$V_0=-r_0V_0'.$$

The latter equation gives the value of the constant a^2, viz.:

$$a^2=\frac{M(r_0)}{r_0}-\int_0^{r_0}\frac{M(r)}{r^2}dr. \tag{42a}$$

Thus the solution of (33), correct to order $\frac{\kappa}{c}$, is

$$r=\left(1+\frac{\kappa}{2c}V_0\right)u, \qquad \sigma(r_0)=0,$$

$$m=M(r_0), \qquad a^2=\frac{M(r_0)}{r_0}-\int_0^{r_0}\frac{M(r)}{r^2}dr.$$

It is seen that the constant m which occurs in the external solution (29) is the mass of the auxiliary density distribution inside the condensation. Also the radius of the condensation does not depend on the radius, R, of the universe. It can therefore be fixed, *a priori*, as the radius of the sphere on which $\sigma(r)$ vanishes, provided, of course, that $r_0 < R$.

II. Application to the Physical Universe

(1) *The Line-element of Physical Space*

We shall now inquire to what extent the metric (29) can be ascribed to the regions external to the condensations observed in the actual universe. The largest of these condensations, the spiral nebulæ, are still minute compared with their distances apart, and therefore, presumably, with the size of the whole universe. And the condensations of stellar type are, in their turn, very small compared with the nebulæ. Hence, from the physical point of view, it would seem, *a priori*, sufficient to use merely the approximate form of (29), obtained by neglecting all terms involving powers of κ/c higher than the first. We should thus be dealing, in effect, with a slightly perturbed form of the metric (5). Then, also, the results of the previous paragraph show that the condensations can have radii independent of R, the initial radius of the universe.

Our starting-point is the theory of the universe which is based on the metric (5). It is known that Lemaître,* working on this basis, has shown that we can obtain the value

$$\frac{\dot{a}}{a}=2\times 10^{-17}\ \text{sec.}^{-1} \tag{49}$$

for the fractional rate of increase of a, from the observed speed of recession of the spiral nebulæ. If, further, it is assumed that

$$\lambda=\frac{c^2}{R^2} \tag{50}$$

and

$$\lambda-\frac{2\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}-\frac{c^2}{a^2R^2}=0, \tag{51}$$

it follows that the universe started expanding from an Einstein universe of radius R and density ρ_e, where

$$R=1{\cdot}01\times 10^{27}\ \text{cm.}, \tag{52}$$

$$\rho_e=1{\cdot}05\times 10^{-6}\ \text{ergs/cm.}^3. \tag{53}$$

Recently Eddington,† working from the side of quantum mechanics, has arrived at these values for R and ρ_e by purely physical theory. He can then deduce the value (49) for $\dot{a}/a$ and hence obtain the velocity of recession of the nebulæ. We shall therefore take the values (52), (53) for R and ρ_e as well established.

Now consider the approximate form of (29), viz. :

$$ds^2=\left(1-\frac{\kappa m}{cr}\right)dt^2-\frac{a^2(t)}{c^2}\left\{\frac{dr^2}{1-\frac{r^2}{R^2}-\frac{\kappa m}{c}\left(\frac{1}{r}-\frac{r}{R^2}\right)}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right\} \tag{54}$$

* G. Lemaître, *M.N.*, 91, 483–490, 1931.

† A. S. Eddington, *Proc. Roy. Soc.*, A, 133, 605, 1931.

and suppose that R is given by (52) and that the unperturbed density is ρ_e. Then the actual density and pressure are, by (13) and (14),

$$\kappa c\rho_{00}=\left(\frac{3\dot{a}^2}{a^2}+\frac{3c^2}{a^2R^2}-\lambda\right)+\frac{\kappa m}{cr}\left(\frac{3\dot{a}^2}{a^2}-\frac{4c^2}{a^2R^2}\right), \tag{55}$$

$$\kappa cp=\left(\lambda-\frac{2\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}-\frac{c^2}{a^2R^2}\right)-\frac{\kappa m}{cr}\left(\frac{2\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}-\frac{c^2}{a^2R^2}\right). \tag{56}$$

Hence

$$\kappa c\rho_e=\frac{3\dot{a}^2}{a^2}+\frac{3c^2}{a^2R^2}-\lambda. \tag{57}$$

To make the metric (54) represent a state of affairs approximating as closely as possible to that given by (5), let us suppose that in regions so distant from the condensation that the terms in $\frac{\kappa m}{cr}$ in (55) and (56) are negligible, the density and pressure are

$$\kappa c\rho_{00}=\kappa c\rho_e=\frac{3\dot{a}^2}{a^2}+\frac{3c^2}{a^2R^2}-\lambda \tag{57a}$$

and

$$\kappa cp=\lambda-\frac{2\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}-\frac{c^2}{a^2R^2}=0. \tag{51}$$

Also we put $\lambda=c^2/R^2$ approximately.* Then the pressure at points near the boundary of the condensation is, by (56),

$$p=\frac{m}{r}\left(\frac{2}{a^2}-1\right)\frac{1}{R^2}, \tag{58}$$

and is therefore negative as soon as $a>\sqrt{2}$.

With regard to the value of $\dot{a}/a$ obtained by using (54) instead of (5), it is easy to show, by making the transformation $R\sin\chi=r-\frac{\kappa m}{c}$ and following Lemaître's analysis, that (49) still holds to a high degree of approximation. The perturbing effect of the condensation is negligible so far as the phenomenon of the recession of the nebulæ is concerned.

There is, however, one capital difference between the states of the matter in the universes represented by (54) and (5) respectively, namely, that in the former the components T_4^1, T_1^4 of the energy tensor do not vanish. Before we can apply (54) to actual condensations we have to calculate the velocity of mass-motion these components involve.

(2) *The Mass-motion*

The equations (16) give, for the co-variant form of T_4^1 or T_1^4,

$$\kappa cT_{14}=\frac{\dot{a}}{a}\frac{\zeta'}{\zeta}=\frac{\dot{a}}{a}\frac{\kappa}{c}\frac{m}{r^2}, \tag{59}$$

* λ will differ from this by an amount proportional to $\frac{m}{R^3}$. See W. H. McCrea and G. C. McVittie, *M.N.*, **91**, 128, 1930, and **92**, 7, 1931.

where we have substituted for ζ from (54) and neglected powers of κ/c greater than the first.

To find what velocity of mass-motion an observer at a point $P(\tau, r_0, \theta_0, \phi_0)$ in the field (54) would measure, we transform to Galilean (Riemannian) co-ordinates at P and calculate the velocity referred to these co-ordinates.

If x_μ denotes any one of the four co-ordinates t, r, θ, ϕ of (54), and ϵ_ν ($\nu = 1, 2, 3, 4$) are the Galilean co-ordinates at P, then *

$$x_\mu = (x_\mu)_0 + \epsilon_\mu - \tfrac{1}{2}\left\{\begin{matrix}\mu\\ \alpha\beta\end{matrix}\right\}_P \epsilon_\alpha \epsilon_\beta - \cdots$$

Hence (54) becomes

$$ds^2 = \zeta(r_0) d\epsilon_4{}^2 - \frac{a^2(\tau)}{c^2}\left\{\frac{d\epsilon_1{}^2}{\mu(r_0)} + r_0{}^2 d\epsilon_2{}^2 + r_0{}^2 \sin^2\theta_0 d\epsilon_3{}^2\right\},$$

where

$$\zeta(r_0) = 1 - \frac{\kappa m}{c r_0} \quad \text{and} \quad \mu(r_0) = 1 - \frac{r_0{}^2}{R^2} - \frac{\kappa m}{c}\left(\frac{1}{r_0} - \frac{r_0}{R^2}\right).$$

If, then, we put

$$\eta_4 = \zeta^{\frac{1}{2}}(r_0)\epsilon_4, \qquad \eta_1 = \frac{a(\tau)\epsilon_1}{\mu^{\frac{1}{2}}(r_0)}, \qquad \eta_2 = r_0 a(\tau)\epsilon_2, \qquad \eta_3 = r_0 a(\tau)\sin\theta_0 \epsilon_3, \quad (60)$$

the line-element (54) becomes

$$ds^2 = d\eta_4{}^2 - \frac{1}{c^2}(d\eta_1{}^2 + d\eta_2{}^2 + d\eta_3{}^2) \quad (61)$$

at the point $P(\tau, r_0, \theta_0, \phi_0)$. In (61) the co-ordinate η_1 is measured along the "radius vector" through P from the centre of the condensation.

Applying the rule for the transformation of tensors, we find that the component $(T_{14})^\times$ of the energy tensor referred to the Galilean system (60) has the value at P,

$$\kappa c (T_{14})^\times = \frac{\dot{a}(\tau)}{a^2(\tau)} \frac{\kappa}{c} \frac{m}{r_0{}^2}\left(1 - \frac{r_0{}^2}{R^2}\right)^{\frac{1}{2}}. \quad (62)$$

Again, the energy tensor can be expressed in any co-ordinate system as

$$T^{\mu\nu} = \rho_{000} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - p g^{\mu\nu},$$

where

$$\rho_{000} = \rho_{00} + p.$$

Hence, referred to the Galilean system (60), the non-zero components are

$$(T_4{}^4)^\times = \frac{\rho_{000}}{1 - u^2/c^2} - p, \quad (63)$$

$$(T_2{}^2)^\times = (T_3{}^3)^\times = -p, \quad (64)$$

$$(T_1{}^1)^\times = -\frac{\rho_{000}\left(\frac{u}{c}\right)^2}{1 - u^2/c^2} - p, \quad (65)$$

$$(T_{14})^\times = -\frac{\rho_{000} u/c^2}{1 - u^2/c^2}, \quad (66)$$

* L. P. Eisenhart, *Riemannian Geometry*, 1926, chap. ii, § 18.

where we have put $u = \frac{d\eta_1}{d\eta_4}$, so that u is the velocity of mass-motion relative to the Galilean co-ordinate system.

Put $q_0 = r_0 a(\tau)$, so that q_0 is the distance of P at the instant τ from the centre of the condensation. Then, equating (62) and (66), we obtain

$$\frac{\rho_{000} u}{1 - \left(\frac{u}{c}\right)^2} = -\frac{\dot{a}(\tau)}{a(\tau)} \frac{m_0}{{q_0}^2} \left(1 - \frac{{q_0}^2}{a^2 R^2}\right)^{\frac{1}{2}}, \tag{67}$$

where $m_0 = m a(\tau)$ and is the mass of the auxiliary density distribution of the condensation as it would be measured by an observer using the co-ordinate q_0 at the instant τ.

For our purpose the term in ${q_0}^2/R^2$ can be neglected, and (67) becomes

$$\frac{\rho_{000} u}{1 - u^2/c^2} = -\frac{\dot{a}(\tau)}{a(\tau)} \frac{m_0}{{q_0}^2}. \tag{68}$$

This formula shows that in an expanding universe, *i.e.* one in which $\dot{a}/a$ is positive, u must be negative. This means that the matter outside the condensation is flowing as a whole towards the centre of the condensation.

(3) *The Isotropy of Pressure*

Let us suppose that the pressure near the condensation given by (58) is entirely due to the random motion of the particles constituting the cosmic cloud in the universe. Let v be the molecular velocity ; then

$$p = \tfrac{1}{3} \rho_{00} \left(\frac{v}{c}\right)^2. \tag{69}$$

It follows from the values (63) to (66) for the energy tensor that we can speak of an *isotropic* pressure outside the condensation only so long as the terms in $\left(\frac{u}{c}\right)^2$ in $(T_1{}^1)^\times$ and $(T_4{}^4)^\times$ are small compared with ρ_{00} and p. To make this precise, we shall say that the pressure is isotropic until

$$\left(\frac{v}{c}\right)^2 = -\frac{u}{c}. \tag{70}$$

This means that our solution (24) or (29) is supposed to break down when (70) ceases to hold, *i.e.* when the process of condensation is sufficiently advanced.

(4) *Numerical Estimates*

The considerations of the last paragraph enable us to say at once that (29) could not be the metric of the universe in its present condition. For at the present time

$$\frac{\dot{a}}{a} = 2 \times 10^{-17} \text{ sec.}^{-1}.$$

Suppose then we calculate from (68) the velocity of mass-motion which ought to be occurring at the surface of (*a*) the Sun, (*b*) a spiral nebula.

(*a*) In this case

$$m_0 = 10^{54} \text{ ergs}, \qquad q_0 = 7 \times 10^{10} \text{ cm}.$$

Hence

$$\frac{\rho_{000}\frac{u}{c}}{1-\left(\frac{u}{c}\right)^2} = -\tfrac{2}{5}10^{-3}.$$

Since ρ_{000} is at most of order 10^{-8} ergs/cm.^{3}, this gives a velocity $u \doteqdot -c(1-10^{-5})$ cm./sec. towards the centre of the Sun. Such a velocity is certainly not observed.

(*b*) Here we take the values of the mass and radius of an average spiral nebula given by Jeans,* viz. :

$$m_0 = 4 \times 10^9 \times \odot = 4 \times 10^{63} \text{ ergs}, \qquad q_0 = 4 \times 10^{20} \text{ cm}.$$

Hence

$$\frac{\rho_{000}\frac{u}{c}}{1-(u/c)^2} = -4 \times 10^{-6},$$

so that with

$$\rho_{000} = 10^{-8} \text{ ergs/cm.}^3$$

we obtain

$$u = -c(1-10^{-3}) \text{ cm./sec.}$$

which is certainly not observed.

We conclude that long before the condensations of nebular or stellar type had reached their present dimensions the solution (29) based on the isotropy of pressure had broken down.

In the foregoing we have assumed that m_0, q_0 and the value of $\dot{a}/a$ are all known, and then deduced u. We shall now carry out the converse process, namely, we shall suppose m_0, q_0 given and then calculate the maximum value of $\dot{a}/a$ consistent with isotropy of pressure. We shall suppose that isotropy of pressure ceases to hold when u/c is numerically greater than v^2/c^2.

We have then from (58) and (69),

$$p = \tfrac{1}{3}\rho_{00}\frac{v^2}{c^2} = \frac{m_0}{q_0 R^2}\left(\frac{2}{a^2} - 1\right),$$

whence

$$\frac{v^2}{c^2} = \frac{3}{\rho_{00}}\frac{m_0}{q_0 R^2}\left(\frac{2}{a^2} - 1\right). \tag{71}$$

This expression is also the maximum possible value of $-(u/c)$ by our assumption. Hence the corresponding value of $\dot{a}/a$ is, by (68) and (71),

$$\frac{\dot{a}}{a} = \rho_{000}\left(-\frac{u}{c}\right)\frac{cq_0^2}{m_0} = 3\left(\frac{2}{a^2} - 1\right)\frac{cq_0}{R^2}, \tag{72}$$

where we have neglected terms in u^2/c^2 and put $\rho_{000} = \rho_{00}$.

Let us now apply (71) and (72) to a condensation of mass equal to that

* J. H. Jeans, *Astronomy and Cosmogony*, p. 335, 1928.

of an average spiral nebula and calculate the values of v and $\dot{a}/a$ for different values of q_0, assuming that

$$R = 1{\cdot}01 \times 10^{27} \text{ cm.} \quad \text{and} \quad \rho_{00} = \rho_e = 10^{-6} \text{ ergs/cm.}^3.$$

We obtain the following table :—

$m_0 = 4 \times 10^{63}$ ergs

q_0	v	$\dot{a}/a$	a	$\bar{\sigma} = m_0/q_0^3$
10^{25}	$1{\cdot}1 \times 10^6$	10^{-18}	$1{\cdot}1$	4×10^{-12}
10^{24}	$3{\cdot}3 \times 10^6$	10^{-19}	$1 + 10^{-2}$	4×10^{-9}
4×10^{23}	$5{\cdot}5 \times 10^6$	4×10^{-20}	$1 + 10^{-3}$	6×10^{-8}
2×10^{23}	$7{\cdot}3 \times 10^6$	2×10^{-20}	$1 + 6 \times 10^{-4}$	5×10^{-7}
10^{23}	$1{\cdot}2 \times 10^7$	10^{-20}	$1 + 3 \times 10^{-4}$	4×10^{-6}
10^{22}	$3{\cdot}3 \times 10^7$	10^{-21}	$1 + 10^{-4}$	4×10^{-3}
10^{21}	$1{\cdot}1 \times 10^8$	10^{-22}	$1 + 10^{-5}$	4

The values of a given in the fourth column are calculated from the formula

$$\frac{\dot{a}}{a} = \frac{c}{R\sqrt{3}} \sqrt{\left(1 + \frac{2}{a^3} - \frac{3}{a^2}\right)},$$

which is the first integral of equation (51) with $\lambda = c^2/R^2$. The fifth column gives the average auxiliary density, m_0/q_0^3, inside the condensation.

It is easy to show that the maximum radius for the given m_0 is of the order of 2×10^{23} cm. For we have to remember that the actual density *inside* the condensation is not $\sigma(r)$, but

$$\kappa c \bar{\rho}_{00} = \frac{3\dot{a}^2}{a^2} \frac{1}{\zeta} + \frac{c^2}{a^2}\left\{\frac{1}{r^2} - \frac{\mu'}{r} - \frac{\mu}{r^2}\right\} - \lambda.$$

Putting $\mu = 1 - \dfrac{r^2}{R^2} + \dfrac{\kappa}{c} r V'$ $\left(\text{obtained from (36) by neglecting terms in } \dfrac{\kappa}{cR^2}\right)$, we obtain for the internal density,

$$\kappa c \bar{\rho}_{00} = \left(\frac{3\dot{a}^2}{a^2} + \frac{3c^2}{a^2 R^2} - \lambda\right) + \frac{\kappa c}{a^2}\left(\sigma(r) + \frac{3\dot{a}^2}{a^2} V(r)\right),$$

so that by (57),

$$\kappa c \bar{\rho}_{00} = \kappa c \rho_e + \frac{\kappa c \sigma(r)}{a^2} + \frac{3\dot{a}^2}{a^2} V(r).$$

Hence, approximately, since $\dot{a}^2/a^2$ is very small and a^2 almost unity,

$$\bar{\rho}_{00} = \rho_e + \sigma(r).$$

Although we cannot say what the value of $\sigma(r)$ is at any point, its average value gives us, at any rate, some guidance as to its magnitude. Hence we have that the actual mass inside radius q_0 is of the order of

$$M_0 = q_0^3(\rho_e + \bar{\sigma}).$$

It follows by inspection of the table that $M_0 >> m_0$ so long as q_0 is greater than 2×10^{23} cm. Hence a condensation of mass approximately that of a spiral nebula could not have a radius greater than this, with our adopted values of R and ρ_e.

The third column now shows that a condensation of radius 2×10^{23} cm. would cease to satisfy the condition of isotropy of pressure as soon as the fractional rate of expansion of the universe reached the value

$$\frac{\dot{\alpha}}{\alpha} = 2 \times 10^{-20} \text{ sec.}^{-1}. \tag{73}$$

If we look upon the moment when our solution (29) breaks down, as the beginning of the formation proper of the nebula, we can say that the fractional rate of expansion of the universe was then given by (73). This enables us to calculate how long it will take for the universe to reach the present rate of expansion, viz.:

$$\frac{\dot{\alpha}}{\alpha} = 2 \times 10^{-17} \text{ sec.}^{-1}, \tag{74}$$

provided we assume that the function α *is given by* (51) *even after* (29) *has broken down.* The value (74) of $\dot{\alpha}/\alpha$ is first attained when $\alpha = 1{\cdot}5$ approximately. It follows from Lemaître's solution * of (51) that the time-lapse from the value (73) of $\dot{\alpha}/\alpha$ to (74) is of the order of 10^{10} years. That is to say, from the moment when the nebula began to collapse till the time when the universe had reached a condition very similar to its present one, some 10^{10} years must have elapsed.

It seems improbable that the solution (29) could apply to condensations of lesser radius than that which we have discussed. From the table it appears that these lesser radii would involve molecular velocities outside the condensation which are $> 10^7$ cm./sec. It seems unlikely that such a high state of disturbance could have occurred so soon after departure from the equilibrium condition ($v = 0$).

Our general conclusion is that if a condensation of mass 4×10^{63} ergs and radius 2×10^{23} cm. formed in an Einstein universe of radius 10^{27} cm., density 10^{-6} ergs/cm.^{3}, and expansion set in, the equilibrium would be quasi-stable till the expansion reached the fractional rate of 2×10^{-20} sec.$^{-1}$. After this a rapid collapse of the condensation would presumably occur until some state of equilibrium corresponding to high condensation in comparatively empty space had been attained.

It is interesting to compare these results with those of Jeans,† obtained on the assumption that (*a*) space is Euclidean and infinite in extent; (*b*) $v = 10^5$ cm./sec.; (*c*) the medium is of density 10^{-9} ergs/cm.^{3}. Then gravitational instability would lead to the expectation that the initial radii of condensations of the mass of spiral nebulæ would be of the order of 9×10^{23} cm., and that the time of collapse to the present radius would be 6×10^{10} years. The present theory seems to indicate that the initial radii, when the curvature of space is allowed for, would be about one-quarter as great as those calculated by Jeans, and that the time-scale is nearer to 10^{10} than to 10^{11} years.

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* G. Lemaître, *loc. cit.*, equation (30).

† J. H. Jeans, *loc. cit.*, chap. xiii, §§ 313 *et seq.*

PHOTOGRAPHIC PROPER MOTIONS OF STARS IN TWO REGIONS.

Violet F. White, M.Sc.

This paper investigates the proper motions of stars in two photographic regions whose approximate centres are :

	R.A.		Dec.	
	h	m	°	′
I. M 27. Dumb-bell nebula	19	55	22	30
II. 52 Cygni	20	43	30	30

The photographs were taken with the 24-inch Wilson reflector, focal length 10 feet 2 inches, now at the University of London Observatory. Plates were obtained in 1894 and 1899 by Mr. W. E. Wilson, and compared with those taken, through the glass, in 1931 by Mr. C. C. L. Gregory, the times of exposure being about 30 minutes.

The measuring machine used * was similar to the Cambridge measuring machine described in *M.N.*, 61, 444, 1901, and the plates were put film to film in the machine with corresponding star images slightly displaced.

Proper motions μ_x, μ_y were deduced from the equations

$$\mu_x = M_x + a + bx + cy,$$
$$\mu_y = M_y + d + ex + fy, \qquad (1)$$

where M_x, M_y are the components of the small distances between corresponding images, and x, y are the co-ordinates of the stars (correct to 0·005 cm. only) referred to the plate centre ; x and y being measured in the directions of increasing right ascension and declination respectively.

The unit of the measuring scale is 0·00005 cm., which is the unit used throughout ; 30 units correspond to 1 second of arc.

Thirty-six of the faintest stars, of photographic magnitudes 13 and 14, for which μ_x, μ_y were assumed to be zero, were used to determine the plate constants, a, b, c, d, e, f. A preliminary graph showed which of these stars were to be rejected on account of appreciable proper motion. A least-squares solution of equations (1), and the method of Christie and Dyson, gave practically the same values for the plate constants.

Region I.—Interval between plates = 37 years.

A diagram was plotted giving the proper motions of the stars in this region as determined from equations (1), *i.e.* relative to the mean motion of the comparison stars as origin, *O*. To determine the absolute proper motions, the stars were considered in 2 groups of mean magnitude (estimated

* Lent by Professor A. Fowler, F.R.S.

H. S.) 10 and 12 respectively, and the mean relative proper motion $\bar{\mu}_x$, $\bar{\mu}_y$ of each group was compared with the mean parallax of each group calculated from

$$P_x = H_m \sin \lambda \sin \chi,$$
$$P_y = H_m \sin \lambda \cos \chi.$$

λ is the angular distance of the plate centre from the solar apex, χ is the position angle of the apex at the plate centre, and H_m is the secular parallax, values for which are taken from *Groningen Publications*, No. 29. The correction to be added to the relative proper motions to convert them into absolute proper motions is then $P_x - \bar{\mu}_x$, $P_y - \bar{\mu}_y$; or A, the origin of absolute proper motions, has co-ordinates $\bar{\mu}_x - P_x$, $\bar{\mu}_y - P_y$, referred to O. The following results were obtained :—

TABLE I

Mag.	n	H_m	P_x	$\bar{\mu}_x$	$\bar{\mu}_x - P_x$	P_y	$\bar{\mu}_y$	$\bar{\mu}_y - P_y$
		"						
10	83	0·013	5·8	−5·5	−11·3	−3·4	2·4	5·8
12	89	0·007	3·1	−2·3	− 5·4	−1·8	3·0	4·8
Total	172	0·010	4·4	−3·8	− 8·2	−2·6	2·8	5·4

where n is the number of stars in each magnitude group.

The centennial proper motions of 2 stars given in *Eigenbewegungs-Lexikon* compared with the measured proper motions as follows :—

B.D.	Mag.	Catalogue		Measured	
		μ_α	μ_δ	μ_α	μ_δ
°		s	"	s	"
22·3887	7·8	0·00	−25·1	0·00	−22·9
21·4025	8·5	0·50	0·0	0·41	− 0·6

These require a mean correction (6, −13) to be added to the measured relative proper motions to convert them to absolute proper motions, whereas Table I gives the value (8·2, −5·4).

The systematic motions of the stars were then determined in the manner originated by Sir A. S. Eddington. With A as origin, sectors $\theta \pm 5°$ (θ = position angle = 0°, 10°, 20° . . . 350°) were taken, the number of stars in each sector was counted, and with this number as radius vector, r, a curve $r = f(\theta)$ was drawn. The values of r and θ are given in Table II. (2. stands for $2\frac{1}{2}$, and so on.)

Fig. 1 shows the resulting curve for all the stars in Region I, and figs. 2 and 3 are the curves for the separate magnitude groups. The curves show the tendency of the stars to move in particular directions, as already found for faint stars in various regions by W. M. Smart (*M.N.*, **84**, 3, 123, 481, 1923–24; **85**, 433, 1925; **87**, 122, 1927) and ten Bruggencate (*B.A.N.*, **3**, No. 85, 39, 1925).

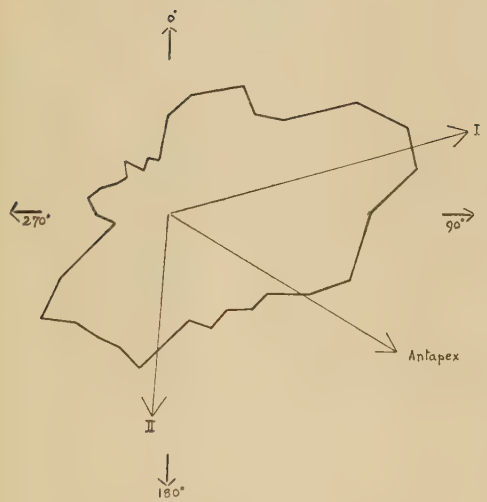


FIG. 1.—*Region I.* 172 *stars.*

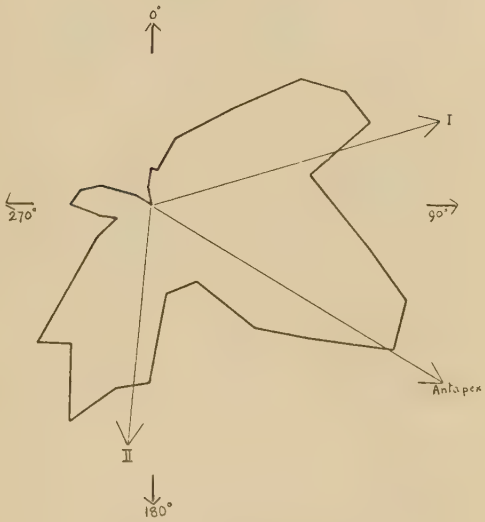


FIG. 2.—*Region* I_{10}. 83 *stars. Mean mag.* 10.

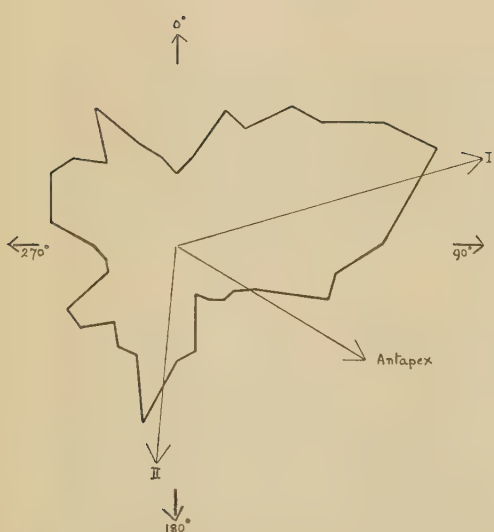


FIG. 3.—*Region* I_{12}. 89 *stars. Mean mag.* 12.

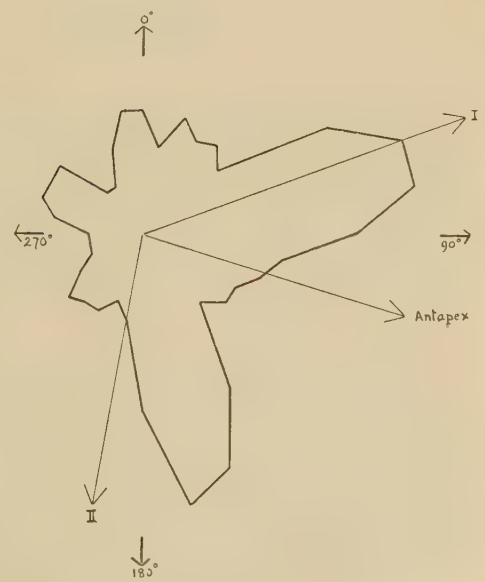


FIG. 4.—*Region II.* 86 *stars.*

TABLE II

	Region					Region			
	I	I_{10}	I_{12}	II		I	I_{10}	I_{12}	II
θ	r				θ	r			
°					°				
0	2.	0	0	2	180	4.	2.	4	2.
10	5	1	4	1	190	6	6	1	2.
20	6	2	1	2	200	8	3	5	2
30	3.	2	3	4	210	3	5	0.	1.
40	6.	2	3.	0	220	6	3	1	2.
50	4.	3	3.	2.	230	8	2	4	1
60	5.	6	3.	3	240	4.	1	2	2
70	14.	3.	7.	6.	250	1.	1	1	1
80	8.	3.	5.	6	260	2	2	1	0
90	5	2	1.	3	270	2.	1	2	2
100	9	4.	5	3	280	3.	2	1.	1
110	7.	7	3	2	290	3	1	3.	2
120	5.	4.	1	2	300	1.	0	2.	3
130	6.	5	1	2.	310	2	0	2	1
140	3.	1.	2	2.	320	2.	0	3	2
150	5	2.	1	1	330	3	0	1	0
160	4	1	1.	6	340	2	0	5	1
170	4.	1.	1.	6	350	1.	1	0	4

The arrows are drawn in directions calculated from the following assumed positions of the solar antapex and apices of Drifts I and II :—

	R.A.	Dec.
	°	°
Solar antapex	90	−34
Drift I	90	−12
,, II	288	−64

The proper motions are the means of 6 measurements, 3 made with the plates in one position, and 3 with them turned through 180°. The probable error of the proper motions, calculated from the discordances of these measurements, is ± 0″·36 per century in both right ascension and declination, for the mean of the 6 measurements. The probable error calculated from the residuals of the comparison stars is ± 0″·37 in right ascension and ± 0″·43 in declination.

Region II.—Interval between plates = 32 years.

All the stars measured are within the dense region between N.G.C. 6960 and 6962, the veiled nebulæ in Cygnus.

The proper motions analysed as in Region I are given in Table III, and the curve in fig. 4 was deduced from the absolute proper motions.

Thirteen stars in this region, mean magnitude 8·5, have proper motions determined from meridian observations (*Greenwich Catalogue of Stars*, 1910), but it was difficult to measure sufficiently accurately the images on the plate of these brighter stars. However, the measurements indicated a correction (22·1, −25·3) to be added to the relative proper motions, compared with the value (15·2, −5·9) given by Table III.

Table III

Mag.	n	H_m	P_x	$\bar{\mu}_x$	$\bar{\mu}_x - P_x$	P_y	$\bar{\mu}_y$	$\bar{\mu}_y - P_y$
		″						
10	46	0·013	6·7	− 8·4	−15·1	−2·0	−0·2	1·8
12	40	0·007	3·6	−11·8	−15·4	−1·1	9·4	10·5
Total	86	0·010	5·2	−10·0	−15·2	−1·6	4·3	5·9

The images on the 1899 plate were not so well defined as on the other plates ; consequently the probable error of the proper motions is larger than for Region I. The probable error calculated from the discordances of 6 measurements is ± 0″·64 per century, in both right ascension and declination, for the mean of the 6 measurements.

The proper motions in both regions were mostly less than 4″ per century.

I gratefully acknowledge the suggestions of Mr. C. C. L. Gregory, who was always ready to help during the preparation of this paper.

THE NAUTICAL ALMANAC OFFICE BURROUGHS MACHINE.

L. J. Comrie, M.A., Ph.D. (Plates 5, 6.)

Introduction.—During the past six years the calculations done in H.M. Nautical Almanac Office have been completely mechanised. Not a single logarithm is now used. The older generation has been succeeded by one which knows only how to produce figures mechanically. The policy of the Office is not to design special machines, but rather to adapt existing commercial machines to its requirements. A specially designed machine is naturally expensive, and is usually the product of a single brain rather than one that has been evolved by many engineering minds. It is liable to develop mechanical faults, and requires special service. These disadvantages do not exist to the same extent in commercial machines, especially those that are made in large quantities to meet modern accounting needs. To show how these existing and purchasable machines can be used for scientific computations really confers a greater benefit on science than inventing or designing a legion of special machines.

Three years ago a Burroughs Class 11 machine was installed to perform what might be called mechanical integration—here the building up of a function from its known finite second differences. In other words, the machine is really a difference engine, realising the ambitions of Charles Babbage, to whom, it may be recalled, the first Gold Medal of our Society was awarded in 1823 for "his invention of an engine for calculating mathematical and astronomical tables." Brief surveys of attempts during the past hundred years to construct difference engines have been given elsewhere.*

Keyboard.—The Class 11 machine is an adding-subtracting listing

* *M.N.*, **88**, 450 (1928 March), and L. J. Comrie, "Modern Babbage Machines," in *Transactions of the Office Machinery Users' Association*, 1932.

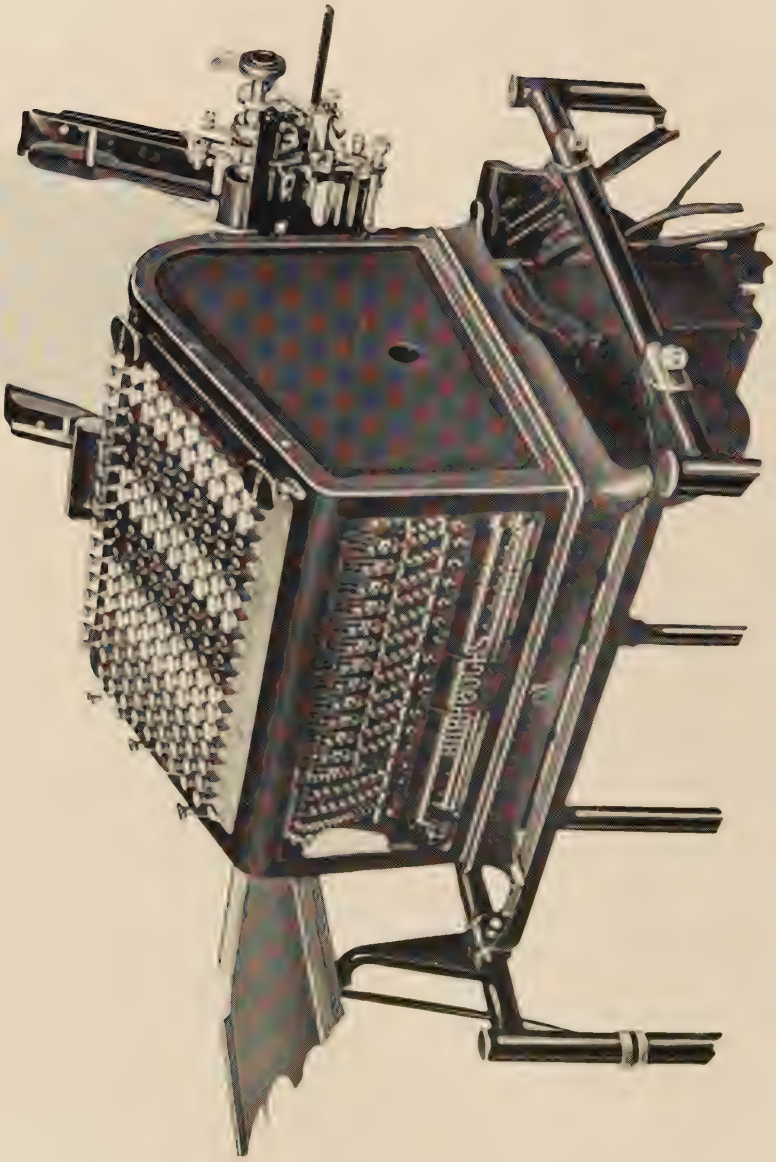
machine, whose commercial function is ledger-posting. The complete machine is shown in Plate 5, and the keyboard in Plate 6. The latter, through which numbers are conveyed to the printing and adding mechanisms, contains 17 columns. Of these the right-hand 13 only are associated with adding mechanism. This part of the keyboard is not flexible (although it can be made so if desired), *i.e.* if a key is depressed in one column the pressing of a second key in the same column will not release the first key and set the second ; the release must be effected by a special release key at the top of the column, or by the master release key, known as the "error key," on the right of the keyboard. There is no key numbered 0 as the machine automatically prints all ciphers to the right of the first figure set on the keyboard. Numbers set on the keyboard are printed and added by a touch of the motor-bar on the right, which brings into operation a stop-and-start motor, *i.e.* one that runs only when required. This operation also releases the keys set, unless the "repeat key" on the right is depressed and latched down.

The four left-hand columns on the keyboard are flexible, and control figures that print only. Any figure set on this section is normally printed at every stroke of the motor-bar, but means are available, and will be described later, for preventing the printing of figures set except in certain specified positions of the movable carriage. These keys, like those of the 13 adding columns, tend to be released after each stroke, but may be retained set, not only by the master repeat key already mentioned, but also by a special key on the left of the machine controlling these four columns only. This key is known as the "date-repeat key," because commercial machines usually utilise these columns for printing dates, which remain constant during a whole day's posting. Here the columns are used for printing the arguments of the functions that are being developed. These arguments usually have an increment of unity in the last figure, so that the operator, at some stage in each cycle, depresses in the units column the key above that already set. It has been found helpful to use different type for these four columns, and also to use a bichrome ribbon, so that their figures are printed in red.

The Crossfooter.—The principal adding mechanism is known as the crossfooter, a term that arises from the operation of "footing" or totalling an account, which on these machines is done by adding or subtracting the entries across the account, and printing the total in an adjacent column. When the crossfooter is in operation items set on the keyboard are added into it, and printed at the same time. If, after the last item has been added, an idle or spacing stroke is made, the total of the items added may be printed by depressing the total or sub-total keys on the left of the machine, in conjunction with the motor-bar. The former clears or zeroises the machine, while the latter leaves the total intact on the accumulator wheels.

The manner in which adding is effected is illustrated in fig. 1,* which shows one unit of the machine. The lever A, pivoted in the middle, carries at the left a set of five type-blocks containing figures from 0 to 9, which

* Reproduced by permission of the Royal Society of Edinburgh, from the *Napier Tercentenary Celebration Handbook* (1914).



General View of the Machine.

Return key
Register non-add key
Register totalling or
Sub-total key
Total key
Release keys
Error key
Repeat key
Vertical bar
RETURN
VERTICAL

Keyboard.

move in slots, and are held back by springs. In the course of a printing operation these type-blocks are raised, and finally arrested with the desired figure opposite the centre line of the platen. At this stage a spring-actuated hammer (not shown in the figure) is released and strikes the rear end of the type-block, driving it forward against the type-ribbon and paper. To the right-hand end of A is attached a toothed segmental rack or sector B, with which a numeral or accumulator wheel C can be thrown in or out of mesh. The upper end of this sector moves between two guide-plates D,

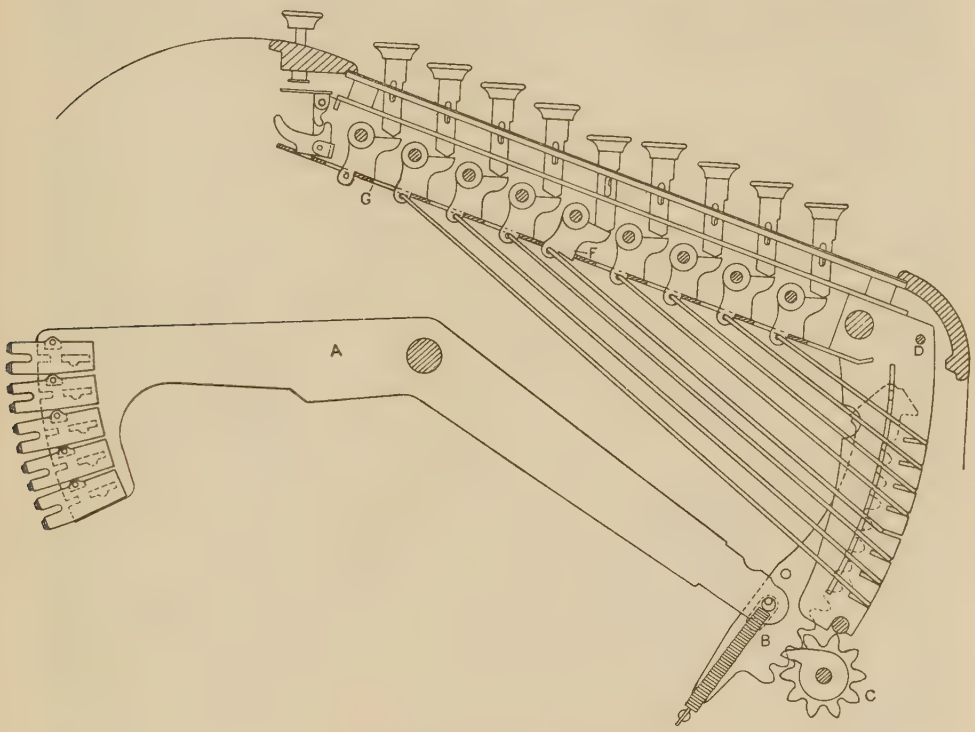


FIG. 1.—*Setting and Adding Mechanism.*

and carries a small nose or projection. The depression of any key, *e.g.* No. 5 as in the diagram, moves a bell-crank F, to which is attached one end of a wire. The other end of this wire is in the form of a hook moving in a slot in the guide-plates D. At the beginning of an adding stroke C is thrown out of mesh, and the sector B is drawn down by a spring until the nose on its upper end, by coming into contact with the wire that has been drawn forward, limits its motion. By this means the type at the other end of A has been positioned for printing, and the printing hammers are released. C is then thrown into mesh, and the sector driven upwards to its normal position, thus rotating C as many teeth as the number set on the keyboard. If the adding process be repeated the accumulator wheel C will be again rotated ; for each complete revolution the adjacent accumulator wheel to the left must be turned one tooth, an operation known as carry-over or tens transmission, and described more fully later.

When the total or the sub-total key is depressed (which cannot be done if any adding key is depressed) C remains in mesh during the downward motion of the sector. This motion, which when adding was limited by the wire positioned by an adding key, is now limited by a pawl (actually the carry-pawl), which prevents the accumulator wheel being rotated beyond its zero position. Thus the type end of the lever A is positioned, and the total printed. This process has rotated the accumulator wheels to their zero positions ; if they are moved out of mesh before the sectors are returned the machine has been cleared and a "total" printed ; if they remain in mesh the wheels are restored to their original positions, and a "sub-total" has been printed. It will be realised, of course, that the printed figures of a total and of a sub-total are the same ; the terms merely distinguish whether the machine has been cleared or not.

The depression of an adding key, besides moving the stop-wire in its slot, withdraws a latch that would otherwise prevent the descent of the sector. It will be noticed that key No. 9 has no wire, as a 9 is added by allowing the sector to descend to its fullest extent. A key depression also locks every other key in the same column. The tail of each key rests on the horizontal arm of a small bell-crank, the other end of which is connected to the stop-wire. As the key is depressed the vertical leg of the bell-crank moves to the left, and carries with it a sliding-plate G, through a slot in which the lower arm of the bell-crank passes, as indicated at F. In the position shown, with key No. 5 depressed, solid metal has been brought under the noses of each of the other bell-cranks, so that it is impossible to depress any other key. The sliding-plate is constantly impelled to the right by a spring, but is prevented by a pawl at its left-hand end from restoring when the pressure on the key is removed. After an item has been printed, the return motion of the machine lifts this pawl, thus allowing the plate to slide back and the depressed key to be raised to its normal position. If, however, the repeat-key is depressed, the pawl is not lifted, and consequently the depressed key remains down.

Subtraction.—Subtraction in the crossfooter is effected by introducing an intermediate wheel between the sector and the ordinary numeral wheel, the two wheels being permanently in mesh with each other. This is shown in the right-hand part of fig. 2. An add-subtract lever, moving in as lot at the left of the keyboard, determines whether the adding or the subtracting wheel is brought into mesh with the sector. It is thus evident that either wheel always contains the complement of the number contained in the other. If a total (or sub-total) is printed with the add-subtract lever in the subtract position, it is the subtracting wheel that is thrown into mesh with the sector, so that the machine prints the complement of the number in the crossfooter. If, for example, the numbers 66 and 88 are added, the machine will print 154 if the lever is in the add position and 9999999999846 if it is in the subtract position. Similarly 66 minus 88 will print 9999999999978 in the add position and 22 in the subtract position. Many applications are made of this facility.

A visible numeral wheel, which may be viewed through the glass front

of the machine, is attached to the crossfooter adding wheel. Practically no use is made of this feature, however, as all results are printed.

The Register.—Another separate adding mechanism, known as the register, forms the distinctive feature of a Class 11 machine. It consists of adding wheels and carry-over mechanism, as shown in fig. 2 ; it has no visible numeral wheels, and the carry-pawls are replaced by wide teeth. By means of its controls it may be thrown into mesh with the sector (as in the diagram) in the same way as the crossfooter wheels, and so can add an item set on the keyboard. As there is no intermediate wheel, it cannot subtract an item from a keyboard setting. There are thus six ways in which

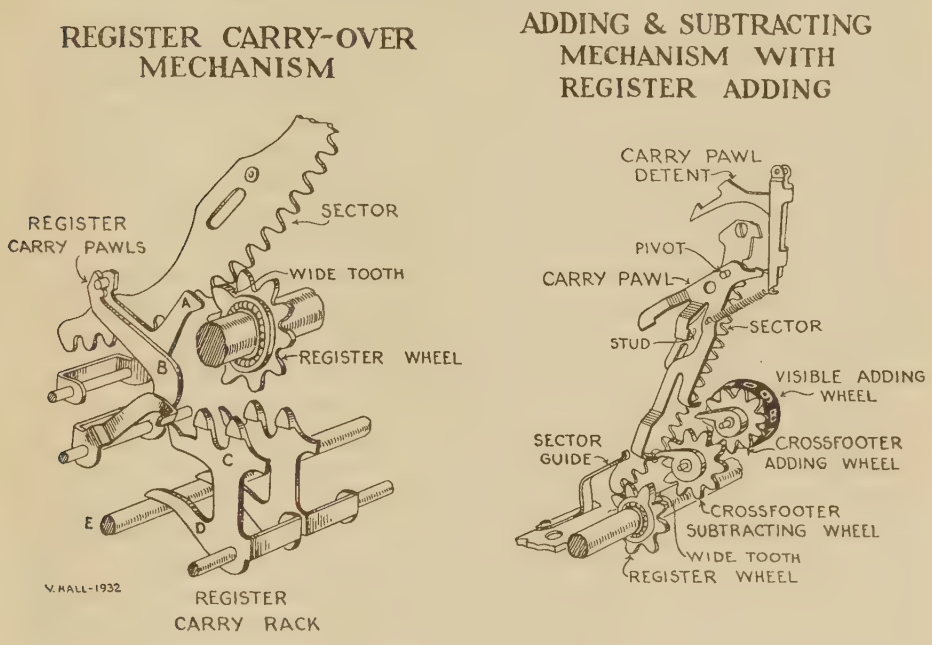


Fig. 2.

Crossfooter	Register
Add	Add
Add	Non-add
Subtract	Add
Subtract	Non-add
Non-add	Add
Non-add	Non-add

the two mechanisms can function, as shown in the accompanying table. The last of these is used in commerce when it is desired to print some non-additive information in the course of a posting, *e.g.* a folio or cheque number. Register totals and sub-totals are taken in the same way as crossfooter totals and sub-totals.

Transfer from one Mechanism to another.—It is the feature now to be described that gives to the machine its possibilities for mechanical integration. It has already been explained that in taking a total (or sub-total) an accumulator wheel is in mesh with the sector on its downward stroke. If, after the sector has been thus positioned, and before the return stroke commences, the accumulator wheel of the other mechanism is thrown into mesh, it is evident that the amount in the first mechanism is transferred to

the second. If the add-subtract lever is in the add position the contents of the crossfooter may be added to those of the register, or *vice versa*. If it is in the subtract position the crossfooter subtracting wheel engages with the sector ; thus the totalling (or sub-totalling) of the register subtracts its contents from those of the crossfooter, while, on the other hand, if the crossfooter be totalled (or sub-totalled), the complement of the number it contains is *added* to the register, thereby effecting indirectly a subtraction from the register of the contents of the crossfooter.

The operator must be able to decide, when totalling either mechanism, whether its contents are to be transferred to the other mechanism or not. In the case of transfer from the crossfooter to the register, specific controls (to be described later) determine, in each column or position of the moving carriage, whether the register is "active" or not, *i.e.* whether it is thrown into mesh with the sector just before the latter commences its return stroke. If it is active, transfer will occur. Transfer (or otherwise) in the reverse direction, *i.e.* from register to crossfooter, is determined by the mode of operating when totalling the register. A "register totalling or sub-totalling lever," moving in a slot at the left of the keyboard, is normally in a middle or neutral position ; it is displaced forward if a register sub-total is desired, and backward for a total. But this movement must be combined with either the sub-total key or the total key and the motor-bar ; if the sub-total key be chosen the contents of the register will not be transferred to the crossfooter, and *vice versa*. The position of the add-subtract lever, as already explained, determines whether the receiving mechanism receives the contents of the other mechanism positively or negatively.

Carry-over.—The methods by which carry-over is accomplished are illustrated in fig. 2. In the case of the crossfooter, as the numeral wheel that is in mesh (*i.e.* either the adding or the subtracting wheel) passes from 9 to 0 its attached carry-pawl comes in contact with a pivoted carry-over pawl to the left, which it displaces backwards. A carry-over pawl detent, attached by a spring to the carry-over pawl, retains it in the displaced position until its function has been performed. On the moving sector is a stud, which, if the carry-over pawl is not displaced, comes into contact with an offset leg on the carry-over pawl, thereby limiting the return movement of the sector when there is no carry-over. If the carry-over pawl is displaced, the sector becomes free to rise another tooth, and in so doing naturally rotates the wheel with which it is meshed. During the next stroke the carry-over pawl is reset before any addition occurs. The total or sub-total keys cannot be depressed unless all the carry-over pawls are set, and it is for this reason that a so-called spacing stroke must be made before a total is taken, unless it should so happen that the last item added did not cause any carry-over.

If the add-subtract lever is reversed, two spacing strokes must be made if it is desired to print a total immediately after the reversal. If, however, any items from the keyboard are entered after the reversal, one spacing stroke only is required before a total.

It will be realised that independent carry-over mechanism must be pro-

vided for the register, and that the latter must be thrown out of mesh with the adding sector before the latter is given the opportunity of rising an extra tooth. The wide tooth on the register wheel displaces the register carry-pawl A, and this in turn, by means of another pawl B, releases the 2-toothed carry-over sector C, which is impelled to the left by a spring, so that its prong D falls into contact with the shaft E. At the end of the adding stroke the register wheel is thrown out of mesh with the adding sector, and into mesh with the carry-over sector. At this stage the shaft E is dropped, and any carry-over sector that has previously been released moves forward, thus rotating the register wheel another tooth. On the next machine operation the register wheel is thrown out of mesh, and the carry-over sectors restored to their normal position by the raising of E. No spacing strokes are necessary before taking a register total or sub-total.

Control from Carriage Position.—In the application of a machine of this nature the operations to be performed occur in cycles, and it is advisable that the operator should be given as little responsibility as possible, beyond that of entering the correct figures on the keyboard, and of performing a few simple mechanical movements. The degree of automatic control that is effected is truly remarkable. The results are printed on paper carried by a moving carriage, in this case $12\frac{1}{4}$ inches wide. The carriage positions (corresponding to columns) that may be assumed are determined by a set-up of stops on a stop-rod. The stop-rod is easily removed from the machine and replaced by another, or it may be stripped, and an entirely different assembly of stops made.

There are five principal types of stop shown on the right of fig. 3. Nos. 12 and 13, which are longer than any of the others, cause a feeding of the paper at each operation of the motor-bar. The number of lines in each feed is determined by an adjusting lever on the right of the carriage, and may be 0, 1 or 2 ; a feed of 0 neutralises the effect of the extra length of these two stops. Stops No. 6 and 13 are seen to be solid, and this has the effect of causing the carriage to " tabulate," or move to its next position, whenever the motor-bar is operated. Nos. 12 and 14 are indented, and allow the carriage to tabulate only when the total or sub-total key has been depressed ; their function is to retain the carriage in position during the taking of spacing strokes, which, as has been explained, are usually necessary before totals can be taken. An S stop prevents all tabulation ; the carriage will leave this stop only in response to the " return-bar " or " return-key," to be described later. Combinations of these stops are available in such a form that either member of the combination can be brought into use at a moment's notice. The illustration shows No. 16, which enables the tabulating effect of a No. 6 or the " wait-for-a-total " effect of a No. 14 to be secured at will, the change being made by rotating the stop through 180°.

Other functions of the machine are controlled by a system of levers (fig. 3) known as the control-arms. The external members of these arms are depressed to varying extents by adjustable control-rolls, carried on a control-rod, which, like the stop-rod, is easily removed and exchanged or

reassembled. In setting-up the machine the positions of the stops are first adjusted to give the desired distances between columns, and then the block carrying the control-rolls is moved laterally until its rolls are centred on the flat portion that may be seen on each control-arm. The arms (except one —the skip) communicate with the inside of the machine, and there exercise their particular functions.

The innermost arm operates on the add-subtract lever. If the arm is not depressed the lever is set to add ; if it is depressed (by a medium-sized

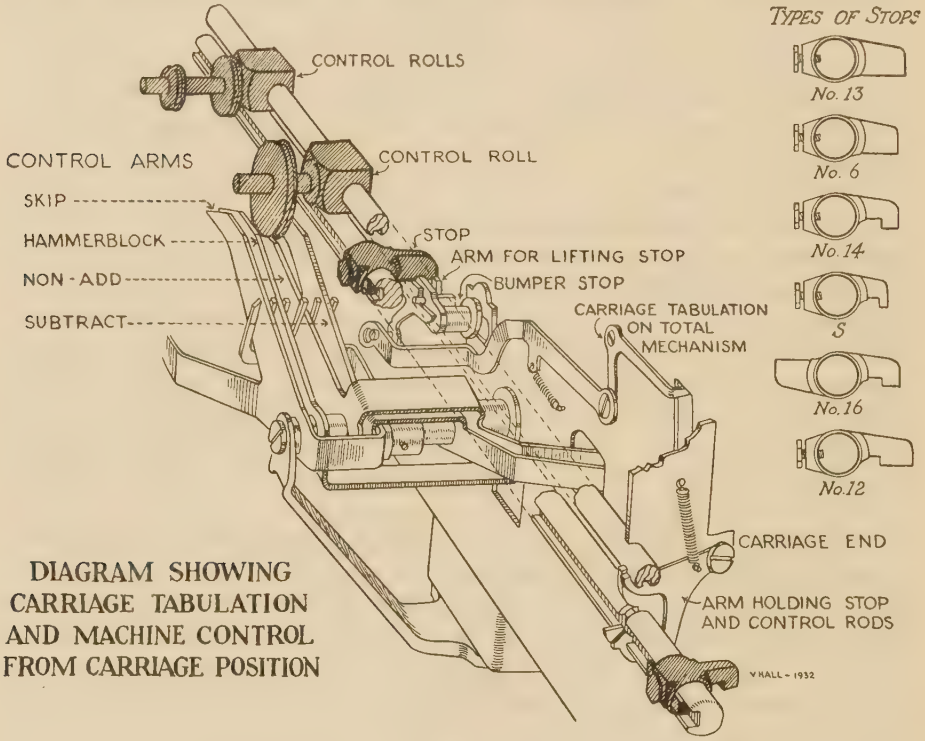


Fig. 3.

roll) the lever is set to subtract. However, the lever is held in position by spring pressure only, and may be reversed by hand at the keyboard if it should be desired to secure a casual reversal of its normal setting. The next arm, known as the non-add, controls the activity of the crossfooter and register. It has four positions as follows :—

No roll.	Both mechanisms active.
Small roll.	Crossfooter active ; register non-adds.
Medium roll.	Neither mechanism active.
Large roll.	Crossfooter non-adds ; register adds.

Thus these two arms enable the machine to select, in any particular carriage position (*i.e.* in any column), any one of the six possible modes of functioning described in the paragraph on the register. On the left of the keyboard are two keys, marked " Non-add " and " Register non-add," which may

be used to secure temporary inactivity of the crossfooter and register respectively. They are restored to their normal position by any operation of the motor-bar, as their purpose is not to secure a continuous effect, which is done by control-rolls, but a casual effect, as for instance in starting a series of operations.

The next arm, known as the hammer-block, operates on the hammers which strike the type. When not depressed all 17 columns are free to print. The effect of varying depressions can be specified, within certain limits, when ordering the machine. Here they suppress either the entire printing, or else the non-additive figures that represent arguments ; thus the latter can remain set on the keyboard (using the date-repeat key), or be altered at will at any stage of a cycle, and yet be allowed to print only in one particular position of the carriage.

The outermost or skip-arm exercises no function when not depressed. If depressed it causes the carriage to move, on leaving the stop with which the skip is associated, not merely to the next stop, but until the skip is broken by an adjustable skip-breaker (not shown in the figure). The carriage is then arrested at the stop next following the point where its skip is broken.

After reaching the end of a line the carriage is returned by a drive from the motor. This return is actuated by a contact made by a movable cam carried on a rack. The return is broken by another movable cam at the other end of the rack. It is thus possible to have a stop-rod with a large number of stops, but to limit the travel of the carriage to a particular portion of the rod by adjusting the cams ; also, as has been seen, stops can be passed over by using the skip feature. A stop may also be eliminated by using a combination stop, of which one member is absent. The carriage return always causes a paper feed of one line.

Occasionally it is desired to return the carriage, not to the beginning of the line, but to some selected intermediate point. This is effected by using the " return-bar," situated above the ordinary motor-bar ; besides exercising all the functions of the latter it causes the carriage to return until the return is broken, not in the usual way by the cam, but by a control-roll operating on a lever associated with the return mechanism (not shown in the figure). When the return has been broken in this way, *i.e.* between two stops, the carriage travels forward once more to the next stop. This feature is known as the " selective column return." If no return-break roll is fitted the return is broken by the cam already mentioned ; this enables a return to be made to the beginning of a line from any desired point without completing the line, and is useful when the termination of the line does not occur at a fixed point. Another possibility is given by the fact that the position of the cam may be so adjusted that the normal return brings the carriage to the beginning of the line, while the return-bar brings it to the second stop of the line. A return key, on the left of the keyboard, is used to produce a casual return to the beginning of the line. The carriage may, of course, be tripped from one stop to the next by hand, or returned by hand if it should be desired (casually) to avoid a feeding of the paper. If it

is thrown back, as it must be for inspection of figures just printed, no printing can take place, and there will be no paper feed on the return.

The "carriage-normal" key, on the right of the keyboard, when depressed and latched down, prevents automatic carriage tabulation, and causes paper feeding of 0, 1 or 2 lines, according to the adjustment of the paper-feed lever. The "vertical bar," below the ordinary motor-bar, combines the functions of the motor-bar and the carriage-normal key. If it is desired to use it solely to prevent tabulation the feeding mechanism is set to feed 0 lines. The feeding mechanism on the carriage return is quite separate, and is unaffected by this adjustment.

The Unlimited Split.—Just behind the middle of the keyboard is an assembly of 17 small levers, protected by a hinged cover, known as the unlimited split, whose purpose is to suppress the printing of unwanted figures. These levers, which are conceived as acting *between* the various columns, may be depressed either fully, or to a half-way position. If a number set on the keyboard is being printed, a split, whether full or half, will suppress all ciphers between the split and the next significant figure. Splits may thus be used to divide the keyboard and printing section into groups, or, if the full capacity of the keyboard is not required, to suppress unwanted ciphers on the right. Thus

Splits					↓						↓		
Column	13	12	11	10	9	8	7	6	5	4	3	2	1
No. on keyboard	0	1	2	3	4	0	0	5	6	7	8	0	0
How printed		1	2	3	4			5	6	7	8		

More elaborate effects are secured when printing from the crossfooter or register. A half-split, as before, suppresses ciphers between the split and the next significant figure. A full-split causes complete suppression of all figures to its right, unless a half-split is set further to the right, in which case the latter acts as a "reviver," causing a group of figures to be printed on its right, but, of course, with the suppression of any preliminary ciphers. Thus

	Half				Full	Half				Full			
Splits	↓				↓	↓				↓			
Column	13	12	11	10	9	8	7	6	5	4	3	2	1
No. in machine	9	0	8	7	6	5	4	0	0	9	8	7	6
How printed	9		8	7	6					9	8		

Between columns 13 and 14, *i.e.* between the adding and non-adding positions, there is always the effect of a half-split.

Mechanical Integration.—It has already been explained that the chief application of the machine has been to the building up of functions from their predetermined second differences.* Specimens are shown in Figs. 4 and 5. In the first of these the function and its differences are arranged in

* The method by which the second differences are predetermined is outside the scope of this paper. See *M.N.*, **88**, 506 (1928 April), and *N.A.*, **1931**, 828; a variant of the method there described is given on p. 537 of this paper.

x	$f(x)$	Δ'	Δ''
1200	2.359253586540		
		62630965	
1201	2.359316217505		21826
		62652791	
1202	2.359378870296		21835
		62674626	
1203	2.359441544922		21847
		62696473	
1204	2.359504241395		21860
		62718333	
1205	2.359566959728		21870
		62740203	
1206	2.359629699931		21881
		62762084	
1207	2.359692462015		21894
		62783978	
1208	2.359755245993		21905
		62805883	
1209	2.359818051876		21915
		62827798	
1210	2.359880879674		

FIG. 4.—*Integration from Second Differences. Ordinary Spacing.*

x	$f(x)$	Δ''	Δ'
1200	2.359253586540		62630965
1201	2.359316217505	21826	62652791
1202	2.359378870296	21835	62674626
1203	2.359441544922	21847	62696473
1204	2.359504241395	21860	62718333
1205	2.359566959728	21870	62740203
1206	2.359629699931	21881	62762084
1207	2.359692462015	21894	62783978
1208	2.359755245993	21905	62805883
1209	2.359818051876	21915	62827798
1210	2.359880879674		

FIG. 5.—*Integration from Second Differences. Adopted Spacing.*

the conventional manner. This arrangement was abandoned after about a year's experience in favour of that illustrated in fig. 5, where the first difference is printed on the right, and the second difference in the middle. This has resulted in greater simplicity of operation. The set-up is as follows :—

Position	1	2	3
Crossfooter	Non-add	Add	Add
Register	Add	Non-add	Add
Print	Cols. 1–17	Cols. 1–13	Cols. 1–13
Stop	14	6	14

In the integration of a second-order equation the constants of integration must be known. Here the initial value of the function and of its first difference must be known. The operation is commenced in position 1 by setting on the keyboard the argument 1200 and initial value 2·359253586540 of the function. A touch of the motor-bar prints these, adding the function value in the register only. The carriage is brought by hand to position 3. The leading value 62630965 of the first difference is then set, and by touching the motor-bar it is printed and added in both register and crossfooter. The former now contains the value of the function for argument 1201. Touching the return-key brings the carriage to position 1, with paper feed. Here the register is sub-totalled, printing the argument 1201 and function 2·359316217505 ; at the same time the carriage tabulates to position 2, in which regular cyclic operations are commenced. The cycle is

1. Change the last figure (or figures) of the argument, and set the second difference.
2. Operate the motor-bar. This prints the second difference, adds it to the previous value of the first difference in the crossfooter, and moves the carriage to position 3.
3. Operate the motor-bar to give the necessary spacing stroke.
4. Sub-total the crossfooter. This prints the augmented first difference, adds it to the previous value of the function, returns the carriage to position 1 and feeds the paper one line.
5. Sub-total the register, without transfer to the crossfooter, *i.e.* using the sub-total key. This prints the argument and function, and the carriage moves to position 2, where the cycle is recommenced.

If the first difference is negative its complement is carried in the crossfooter and printed ; if it changes sign the work proceeds without interruption —the machine simply changes from printing a complement to printing a direct number, or *vice versa*. If the second difference is negative the crossfooter is set (by means of a control-roll) to subtract in position 2. Thus in every case the work of the operator is the same—to set a second difference and complete a uniform cycle. A special control-block carrying an eccentric control-roll is used in position 2, so that the second difference control may be changed from addition to subtraction at a moment's notice.

Illustrative Applications of the Machine.—The mean positions of stars are given by the formula (for right ascension)

$$\alpha = \alpha_0 + at + bt^2 + ct^3,$$

where a is the annual precession, b and c are functions of the secular variation and third term, and t is the time elapsed since the epoch of the catalogue. The interval of t is, in ephemerides, one year. The second differences of α are $2b + 6ct$, and are linear. Hence they are formed one by one on a small calculating machine on a side table, and transferred immediately to the keyboard of the Burroughs. As a check the value of α at the end of the run (say for $t = 20$) is calculated independently. Similar expressions in powers of time for mean longitudes, longitudes of node, perigee, etc., are handled in the same way.

It will be realised that the use of the machine is by no means limited to mechanical integration. A supply of additional stops and rolls is kept, so that any desired set-up may be assembled at short notice. On one occasion it was desired to check the consistency *inter se* of reputed values of e^x, e^{-x}, $\sinh x$ and $\cosh x$. Since $\cosh x \pm \sinh x = e^{\pm x}$, the set-up was

Position	1	2	3	4
Crossfooter	Subtract	Add	Add	Non-add
Register	Add	Add	Non-add	Add
Stop	6	6	14	14
Operation	Set $\sinh x$	Set $\cosh x$	Total crossfooter	Total register

The quantities printed in columns 3 and 4 were then proof-read against e^{-x} and e^x respectively.

An every-day statistical problem consists in determining, from observed frequencies f at equidistant abscissæ x, the moments $[f]$, $[fx]$, $[fx^2]$, required in finding means and standard deviations, and often $[fx^3]$ and $[fx^4]$, required in finding the Pearson functions β_1 and β_2. To explain the principle involved consider the evaluation of fx in the following simple case.

x	f	fx
4	17	68
3	19	57
2	24	48
1	31	31
	$[f] = 91$	$204 = [fx]$

Now suppose the values of f to be arranged in descending order of x, and two summations to be performed on them.

x	f	First Summation	Second Summation
4	17	17	17
3	19	$36 = 17 + 19$	$53 = 2(17) + 19$
2	24	$60 = 17 + 19 + 24$	$113 = 3(17) + 2(19) + 24$
1	31	$91 = 17 + 19 + 24 + 31$	$204 = 4(17) + 3(19) + 2(24) + 31$

The last line of the first summation is seen to be $[f]$ and the last line of the second summation to be $[fx]$. The reason for arranging the values of f in descending order of x is quite obvious. The real mathematician will, of course, replace the above exposition by an algebraical one.

The process of double summation is exactly that already described as mechanical integration, so that the set-up is

Position	1	2	3
Crossfooter	Add	Add	Non-add
Register	Non-add	Add	Add
Print	Cols. 1–17	Cols. 1–13	Cols. 1–13
Stop	6	14	14
Operation	Set x, f, fx	Sub-total crossfooter	Sub-total register

x	f	fx				
20	1	20	1	20	1	20
19	1	19	2	39	3	59
18	2	36	4	75	7	134
17			4	75	11	209
16	3	48	7	123	18	332
15	3	45	10	168	28	500
14	5	70	15	238	43	738
13	7	91	22	329	65	1067
12	12	144	34	473	99	1540
11	13	143	47	616	.146	2156
10	17	170	64	786	.210	2942
9	28	252	92	1038	.302	3980
8	24	192	.116	1230	.418	5210
7	43	301	.159	1531	.577	6741
6	53	318	.212	1849	.789	8590
5	57	285	.269	2134	1.058	10724
4	55	220	.324	2354	1.382	13078
3	68	204	.392	2558	1.774	15636
2	83	166	.475	2724	2.249	18360
1	85	85	.560	2809	2.809	21169
			$[f]$	$[fx]$	$[fx]$	$[fx^2]$

FIG. 6.—*Calculation of* $[f]$, $[fx]$ *and* $[fx^2]$.

The result is illustrated in fig. 6, the figures being taken from an example in Pearson's *Tables for Statisticians*, p. lxxv. By means of a split the keyboard is divided into two parts. x is set on the non-additive columns, f on the left of the additive columns, and fx (formed mentally) on the right. The last lines of the two columns printed in the middle position give $[f]$ and $[fx]$, while those in the third position give $[fx]$ and $[fx^2]$. The equality of the two values of $[fx]$, the first of which is the sum of the values of fx set, while the second is the second summation of the values of f set, is a complete proof that the values of fx set were actually the products of x and the values of f set. A misreading of f is possible, and hence an erroneous fx; this is not detectable, unless $[f]$ is known beforehand, as

is usually the case. If for any value of x the frequency f is zero, as for $x = 17$ in the example, there is no setting on the active part of the keyboard, but the line must be completed in the usual way. It may also be observed that frequencies increase as x decreases, a fact that renders the mental formation of fx a simple matter. The illustration covers positive values of x only; negative values are treated in the same way.

If the process be repeated, setting on the keyboard the values in the last column printed, two new summation columns will be produced. The sum of the last two values in the first of these new columns is $[fx^3]$. Call the last three values in the second new column A, B and C; then $[fx^4] = A + 4B + C$. Also $[fx^3] = C - A$. Variations of the method here suggested are possible, and also simplifications if factorial moments are required.

Some Points in Interpolation Technique.—In the end-figure method of interpolation to tenths * the sums in pairs of the second and of the fourth differences are required. Before the advent of the Burroughs this was done by hand, thus leading to occasional errors, which, if large enough, were detected at a later stage. Actually the double fourth differences are the second differences of the double second differences. Hence the double fourth differences are formed by hand, and then the Burroughs is used in the usual way to produce the double second differences, occasional values of which (say 1 in 25) are formed beforehand to serve as checks. These checks will not be satisfied if an error has been made

(*a*) in forming the original third difference;
(*b*) in forming the original fourth difference;
(*c*) in forming the double fourth difference;
(*d*) in setting the double fourth difference on the keyboard.

Thus the process eliminates many sources of error. It may be recalled that multiples of 4000 are rejected from the double second difference. Hence the splits are set so that only the last five figures of the double second difference are printed; from the first two of these figures multiples of 4 are subsequently rejected.

If the third difference is not too large, say less than 30 or 40, its effect on an interpolate may be regarded as negligible. In that case the following simpler procedure may replace the more elaborate technique required when the third and fourth differences are larger. Denote by Δ the differences of the original values, and by δ the differences of the interpolates. For interpolation to tenths δ'' is approximately equal to $\frac{1}{100}\Delta''$. If in the interpolation process we carry two extra and purely fictitious decimals, and work in units of the last of these extra decimals, then δ'' is approximately Δ''. The principle of the method now put forward is to use a value of δ'' that remains constant for nine consecutive values, and then a value so chosen that it corrects the cumulative error arising from using the nine unchanged values of δ'', and also reproduces exactly (*i.e.* with two extra fictitious ciphers)

* See references in footnote on p. 532.

the original values of the function. The value of δ'' required in the interval from f_0 to f_1 is the mean of Δ_0'' and Δ_1'', taken to the nearest even figure ; if this mean naturally ends in an odd figure, it should be lowered numerically if the fourth difference has the same sign as the second, and *vice versa*. Calling this value (which is to be used nine times) $\delta_{\frac{1}{2}}''$, the value in the interval f_1 to f_2 is $\delta_{\frac{3}{2}}''$, and the value opposite f_1, which is used once only, is given by

$$\delta_1'' = \tfrac{1}{2}(\delta_{\frac{1}{2}}'' + \delta_{\frac{3}{2}}'') + 10[\Delta_1'' - \tfrac{1}{2}(\delta_{\frac{1}{2}}'' + \delta_{\frac{3}{2}}'')].$$

The value $\frac{1}{2}(\delta_{\frac{1}{2}}'' + \delta_{\frac{3}{2}}'')$ is necessarily integral, since both $\delta_{\frac{1}{2}}''$ and $\delta_{\frac{3}{2}}''$ are even. The second term on the right is usually zero ; it attains 20 only when the original third difference is appreciable. The method of preparing the required values of δ'' is shown in the accompanying example. The figures in ordinary type are the original second and third differences. The figures in bold type in the first column represent $\delta_{\frac{1}{2}}''$, $\delta_{\frac{3}{2}}''$, etc., while those in the second column are $\frac{1}{2}(\delta_{\frac{1}{2}}'' + \delta_{\frac{3}{2}}'')$, etc. The third column is the supplementary correction (always a multiple of 10) required by the values in the second column. In actual practice the figures shown in bold type are done in coloured ink, so are readily picked out by the Burroughs operator. The leading value of δ' is $10\Delta_{\frac{1}{2}}' - 4{\cdot}5\delta_{\frac{1}{2}}''$.

690		
674	32	
658	**658**	
642	32	
626	**627**	−10
612	29	
597	**598**	−10
584	25	
572	**572**	
558	27	
545	**546**	−10
534	23	
522	**522**	
510	22	
500	**500**	
490	21	
479	**479**	
468	20	
459	**459**	
450	17	
442	**441**	+10
432	18	
424	**423**	+10
414	18	
406	**407**	−10
400	13	
393	**392**	+10
384	17	
376		

This process has the apparent disadvantage of wasting the capacity of two columns. Since the machine adds 13 figures, an 11-figure function can be interpolated ; a function with more than 11 significant figures is not likely to have negligible third differences.

The same principles may be applied to interpolations to other intervals. Thus for interpolation to fifths:

$$\text{Leading } \delta' = 20\Delta_{\frac{1}{2}}' - 2\delta_{\frac{1}{2}}''$$
$$\delta_{\frac{1}{2}}'' = 2(\Delta_0'' + \Delta_1'') \quad \text{(used four times)}$$
$$\delta_1'' = 4(\Delta_1'' - \Delta_1^{\text{iv}}) \quad \text{(used once)}$$
$$\delta_{\frac{3}{2}}'' = 2(\Delta_1'' + \Delta_2'') \quad \text{(used four times)}$$

In building up a function from its second differences any error in setting will be infallibly detected by the resulting failure of the machine to reproduce the next pivotal value. Compensating errors are unknown ; although it is possible to conceive a combination of errors in the second difference that would not affect the pivotal value immediately following, the compensation would not extend to the first difference, so that the error would be detected at the succeeding pivotal value. A skilled operator will make in setting

(especially with small numbers) only a fraction of the errors that would be made by a copyist who wrote the figures by hand.

Rounding-off of Functions.—Most computers who use mechanical aids are familiar with the fact that a series of values produced by a continuous process may be automatically rounded-off by adding 5 to the first rejected decimal. It is evident that, if the first rejected decimal is 0, 1, 2, 3 or 4, this makes it 5, 6, 7, 8 or 9, and does not affect the last figure retained ; if it is 5, 6, 7, 8 or 9, the addition of 5 makes it 0, 1, 2, 3 or 4, and at the same time increases the last figure retained by 1. Considerable application is made of this convenience. Thus it was required to prepare printer's copy for a series of 8-figure tables, which existed as 10-figure tables with second differences, usually of three figures. The initial value of the function was increased by 50, and the function built up anew from its second differences. This is two or three times as rapid, and infinitely more accurate and legible, than copying the figures by hand. The splits could be adjusted to suppress the printing of the ninth and tenth decimals, but actually it is preferable to print them for checking purposes. The printed strip (which is usually in the form of a continuous roll) has to be torn in any case, and is easily torn between the eighth and ninth decimals by laying it against a steel edge, and ripping.

Multiple Copies of Tables.—By using carbon paper additional copies of a table may be made, but such copies are, naturally, not nearly as sharp as the original. An alternative is to print each value of the function several times before developing the next value, afterwards tearing the sheet thus produced into columns, each of which is assigned to one copy of the assembled tables. In this way multiple copies of tables required for internal use have been prepared—for example, six copies of a 7-figure table of tangents for every second of time, used in conversion from longitude and latitude to right ascension and declination.

Checking of Tables by Differencing.—This cannot be done in the usual way, although a modification offers considerable advantages over the ordinary hand process. The end figures of the table are written, and differenced until the end figures of a difference that can be inferred in full are reached. If this difference is greater than the second, the last two figures of the function should be used. The smoothness of the function is tested by this means, and then its values are built up from the difference that has been inferred in full. The copy printed by the machine is then proof-read against the original values ; the proof-reading may, if desired, be done twice. This method is more likely to detect errors in leading figures than ordinary differencing, for, with human frailty, a computer is apt to assume the leading figures of his differences. The following example shows the beginning of the process, and its continuation. Here the building up was done from the fourth difference, so that it was necessary to determine beforehand the points where the second difference changed sign. The first and third differences were treated as positive always. In other words their complement was used when they were negative, thus avoiding any special procedure when they changed sign.

x	$f(x)$	Δ'	Δ''	Δ'''	Δ^{iv}	Δ^{v}	Δ^{vi}
		−	−	+	+	−	
15·47	0·170741713736						
		1160087470					
·48	·169581626266		16132633				
		1176220103		126925			
·49	·168405406163		16005708		1502		
		1192225811		128427		16	
·50	·167213180352		15877281		1486		+3
		1208103092		129913		13	
·51	·166005077260		15747368		1473		
		1223850460		131386			
·52	·164781226800		15615982				
		1239466442					
15·53	0·163541760358						

x	$f(x)$	Δ'	Δ''	Δ'''	Δ^{iv}	Δ^{v}	Δ^{vi}
			−		+	−	
15·47	36						
		30					
·48	66		33				
		97		25			
·49	63		08		1502		
		89		27		16	
·50	52		81		1486		+3
		08		13		13	
·51	60		68		1473		0
		40		86		13	
·52	00		82		1460		−3
		58		46		16	
·53	58		36		1444		+3
		22		90		13	
·54	80		46		1431		0
		76		21		13	
·55	56		25		1418		−3
		51		39		16	
·56	07		86		1402		+2
		65		41		14	
15·57	72		45		1388		−1

Actual Performance of the Machine.—The work on this machine was thoroughly organised in 1930, and a strict record of its performance was kept throughout 1931. Three operators kept the machine running $8\frac{1}{2}$ hours per day, no one operator being on the machine more than 2 hours at a stretch, or more than 4 hours a day. The total number of function values developed and printed was 1,200,000. This involved 6,000,000 motor-bar operations, and the printing of about 30,000,000 figures. It would take a copyist, working ordinary office hours, 7 years to *copy* these figures, *without any calculation*! There can be no doubt that this is a record for any Burroughs machine, due to the peculiar nature of the cyclic operations, namely, a small keyboard setting and the printing of two sub-totals between each setting.

The average speed in integration from second differences is 500 values

an hour, although records of over 1000 values an hour have been attained. A type-ribbon lasts about 10 working days, by which time it has been used for the printing of 1,000,000 figures.

Considering the hard and continual usage to which the machine has been subjected, the amount of mechanical trouble experienced has been slight ; expert and prompt service has always been available when required. As, in general, every tenth value printed is compared with its known value, there has been ample opportunity for detecting incorrect printing ; yet never, during the three years that the machine has been in use, has it been known to print an incorrect figure, *i.e.* one that cannot be accounted for by the known settings and operations.

Later Models.—Burroughs models containing as many as twenty registers are now available. Certain models, containing up to six registers, are similar in external appearance to the machine just described. A number set on the keyboard may be added or subtracted in the crossfooter, and added in any one (but not more than one) selected register. A number in the crossfooter may be transferred to any one register (or *vice versa*) at any one operation ; it may be transferred to other registers by separate operations. The selection of the register that is to be active may be done by means of keys on the keyboard, of a special motor-bar, or of a control-cam or skid acting on a control-arm. Transfer from one register to another can only be effected *via* the crossfooter, which, naturally, must be otherwise clear if used for this purpose. Actually the register wheel shown in fig. 2 is replaced by n (in an n-register machine) register wheels on the same shaft ; the action of selecting a register simply brings one of these wheels into the plane of the sector.

The Class 16 machine, with six registers, could be used for integration from sixth differences, or differencing to the sixth difference. In its present form, *i.e.* as constructed for commercial use, its efficiency factor in these applications would not be so great as that of the Class 11 for integration from second differences, mainly because of the necessity for using the crossfooter as an intermediary between the registers, and because of the large number of spacing strokes required, especially in differencing. The Class 26 machine, also with six registers, requires no spacing stroke if totalling after an additive item, and only one when totalling immediately after a reversal of the add-subtract lever.

Another recent feature that should be studied by anyone contemplating the use of these machines is a cycle mechanism. After the keyboard setting, a single touch of the motor-bar causes the machine to perform a cycle of operations that is governed by controls somewhat similar to those already described.

Conclusion.—It is believed that the discovery of commercial machines that will integrate from finite differences, or difference a function, with printing of the results, has revolutionised the art of table-making. The printer, too, will welcome their introduction, as they are the means of providing him with completely legible copy, in which he can be assured that there will be no author's corrections in proof.

THE 3 : 1 RESONANCE CASE IN PLANETARY MOTION.

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(Communicated by Professor E. W. Brown)

Summary.—This paper contains a discussion of the 3 : 1 case of resonance in the planetary theory. It follows a method given in an unpublished manuscript by Professor Brown, in which the 2 : 1 case is treated in detail. This method is reproduced here so far as is necessary for the discussion in this paper. The 3 : 1 case requires the inclusion of the inclination, and we therefore have *two* principal librating angles. The regions of libration are found separately for the two cases; first, when the inclination is taken to be zero, and secondly, when the eccentricity is zero. These regions are illustrated in two diagrams.

Resonance problems are characterised by the property that $(in - i'n')$ vanishes for small integral values of i and i', where n, n' are the mean motions. One of the peculiarities of the problem is a change from expansions of the disturbing mass to expansions in powers of its square root. A second is the recognition of the fact that resonance phenomena are present over a finite range of values of the arbitrary constants and not merely when one of these takes a particular value.

§ 1. *Statement of the Problem.*—The variables c_i, ω_i satisfying the equations,

$$\frac{dc_i}{dt} = \frac{\partial R}{\partial \omega_i}, \qquad \frac{d\omega_i}{dt} = -\frac{\partial R}{\partial c_i} - \frac{\partial}{\partial c_i}\left(\frac{\mu^2}{2c_1^2}\right), \tag{1}$$

will be used, where c_i, ω_i are defined in terms of the usual elements by the following relationships,

$$\begin{aligned} &c_1 = (\mu a)^{\frac{1}{2}} = c, && \omega_1 = nt + \epsilon = \omega, \\ &c_2 = c(1 - e^2)^{\frac{1}{2}} - c = -\tfrac{1}{2}ce_1^2, && \omega_2 = \varpi, \\ &c_3 = c(1 - e^2)^{\frac{1}{2}}(\cos i - 1) = -c\Gamma_1, && \omega_3 = \theta. \end{aligned} \tag{2}$$

Thus when e is not too great, e and e_1, Γ and Γ_1 are nearly equal.

We shall suppose that the disturbing planet moves in an ellipse in the plane of reference with elements ω_1', ω_2', a', e', where

$$\omega_1' = n't + \epsilon' = \omega', \qquad \omega_2' = \varpi'.$$

For treatment of resonance cases we suppose that the mean motions are nearly in the ratio of two integers, i_0', i_0; so that the terms in the disturbing function R, containing ω, ω' in the combination $p(i_0\omega - i_0'\omega')$ and therefore having the motions $p(i_0 n - i_0' n')$, (p an integer), are those which would give large or infinite coefficients by previous methods. This does not define i_0, i_0' because if n/n' be converted into a continued fraction and if i_0'/i_0 be the first very close convergent, the remaining convergents will still be closer.

The terms containing the latter values of i_0, i_0' will be neglected in view of the fact that a very close convergent is always followed by a convergent having large integers, and that the order of the coefficient with respect to the eccentricities and inclination is the difference of these integers. This definition and limitation permits a sharp separation between the terms under consideration and those of short period. The latter are eliminated by a suitable change of variables which add terms depending on the square of the mass to the Hamiltonian function. Such added terms will be neglected so that the equations for the new variables will be the same as those for the old variables, with the short-period terms neglected. With this understanding we retain the same notation, and equations (1) will be supposed to contain all portions of R which are not of short period.

§ 2. *The Equation for* c.—Replace the variable ω by τ where

$$i_0\omega - i_0'\omega' = i_0\tau. \tag{3}$$

The equations will still remain canonical provided the Hamiltonian function

$$H = \frac{\mu^2}{2c^2} + \frac{i_0'}{i_0}cn' + R \tag{4}$$

is used. Since by the limitations of § 1, ω, ω' occur only in the combination $p(i_0\omega - i_0'\omega')$, the time, which was explicitly present in R only through its presence in ω', will disappear from the Hamiltonian function. Equation (4) therefore reduces to a constant. Define the constants c_0, n_0, and the variable z by the relations,

$$\frac{\mu^2}{{c_0}^3} = n_0 = \frac{i_0'n'}{i_0}, \qquad n = n_0(1+z). \tag{5}$$

The relations

$$c = (\mu a)^{\frac{1}{2}}, \qquad n^2a^3 = \mu,$$

give

$$c = c_0(1+z)^{-\frac{1}{3}}. \tag{6}$$

On making these changes in (4) equated to a constant, we get

$$n_0c_0[\tfrac{1}{2}(1+z)^{\frac{2}{3}} + (1+z)^{-\frac{1}{3}}] + R = \text{constant}.$$

By hypothesis z is small. Expansion in powers of z gives

$$\tfrac{1}{6}z^2 - \tfrac{4}{27}z^3 + \ldots + \frac{R}{n_0c_0} = \text{constant} = \frac{C}{6}. \tag{7}$$

Thus the first power of z has disappeared, a property of fundamental importance.

A first approximation to the value of z is given by

$$z = \pm\left(C - \frac{6R}{n_0c_0}\right)^{\frac{1}{2}}. \tag{8}$$

A case of resonance amongst the mean motions will be defined as one in which z can vanish for some value of the time. In such a case the osculating mean motion and that of the disturbing planet are in the ratio of two integers

for some value of t. In order that this may be possible, values of C must be chosen such that $6R \leqslant n_0 c_0 C$, since R is necessarily positive. Since C is the maximum value which $6R/n_0 c_0$ can have, we can regard it as of the same order of magnitude, and therefore as having the factor m. It thence follows that in resonance cases, z *has* $m^{\frac{1}{2}}$ *as a factor*, and that expansions in powers of $m^{\frac{1}{2}}$, rather than of m, must be made.

§ 3. *The Equation for* τ.—With the definitions of § 2, the equation can be written,

$$\frac{d\tau}{dt} = n_0 z - \frac{\partial R}{\partial c}. \tag{9}$$

Differentiating with respect to t, we get

$$\frac{d^2\tau}{dt^2} = n_0 \frac{dz}{dt} - \frac{\partial^2 R}{\partial c \partial \tau}\left(n_0 z - \frac{\partial R}{\partial c}\right) - \frac{\partial^2 R}{\partial c^2}\frac{dc}{dt} - \sum\left(\frac{\partial^2 R}{\partial c \partial c_i}\frac{dc_i}{dt} + \frac{\partial^2 R}{\partial c \partial \omega_i}\frac{d\omega_i}{dt}\right), \tag{10}$$

where, in the last term, i has the values 2, 3. Since the derivatives of c, c_2, c_3, ω_2, ω_3 contain m as a factor, we get, in neglecting terms in this equation which are factored by m^2,

$$\frac{d^2\tau}{dt^2} = n_0 \frac{dz}{dt} - n_0 z \frac{\partial^2 R}{\partial c \partial \tau}. \tag{11}$$

But the first of equations (1) gives, on substituting for c its value from equation (6),

$$-\tfrac{1}{3} c_0 (1+z)^{-\frac{4}{3}} \frac{dz}{dt} = \frac{\partial R}{\partial \tau}, \tag{12}$$

or, neglecting terms of order m^2,

$$\frac{dz}{dt} = -\frac{3}{c_0}(1 + \tfrac{4}{3} z)\frac{\partial R}{\partial \tau}. \tag{13}$$

Now R is a function of c. If we use the expansion $c = c_0(1 - 1/3 z \ . \ . \ .)$ and expand R in powers of z, we get

$$R = R_0 - \tfrac{1}{3} z \frac{\partial R_0}{\partial c_0}, \tag{14}$$

where R_0 is the value of R when we put c_0 for c. Thence as far as the order $m^{\frac{3}{2}}$,

$$\frac{\partial R}{\partial \tau} = \frac{\partial R_0}{\partial \tau} - \tfrac{1}{3} z \frac{\partial^2 R_0}{\partial c_0 \partial \tau}. \tag{15}$$

Combining this result with equations (5) and (7) we note that $\partial^2 R_0 / \partial c_0 \partial \tau$ disappears and we get as far as $m^{\frac{3}{2}}$,

$$\frac{d^2\tau}{dt^2} = -3\frac{n_0}{c_0}(1 + \tfrac{4}{3} z)\frac{\partial R_0}{\partial \tau}, \tag{16}$$

in which to the same order we can put

$$z = \left(C - \frac{6R_0}{n_0 c_0}\right)^{\frac{1}{2}} = \frac{1}{n_0}\frac{d\tau}{dt}. \tag{17}$$

Thus the variable c has been eliminated and we have to the same order,

$$\frac{d^2\tau}{dt^2}\left(1-\frac{4}{3}\cdot\frac{1}{n_0}\cdot\frac{d\tau}{dt}\right)=-3\frac{n_0}{c_0}\frac{\partial R_0}{\partial\tau}. \tag{18}$$

It is recalled that the variables c_2, c_3, ω_2, ω_3 are still present in the right-hand member.

§ 4. In the preceding investigations we made use of an integral of the equations to eliminate c, and to get an equation for τ in a simplified form. We can obtain another relation which becomes an integral of the equations when the eccentricity of the disturbing planet is neglected.

The disturbing function contains the angles ω, ω', ω_2, ω_3, ϖ', only in the form of their differences, so that

$$\frac{\partial R}{\partial\omega}+\frac{\partial R}{\partial\omega'}+\frac{\partial R}{\partial\omega_2}+\frac{\partial R}{\partial\omega_3}+\frac{\partial R}{\partial\varpi'}=0. \tag{19}$$

By hypothesis R contains the angles ω, ω' only in the form $p(i_0\omega-i_0'\omega')$, so that $i_0'\partial R/\partial\omega+i_0\partial R/\partial\omega'=0$. The change from ω to τ gives $\partial R/\partial\omega=\partial R/\partial\tau$. Hence we have

$$\left(1-\frac{i_0'}{i_0}\right)\frac{\partial R}{\partial\tau}+\frac{\partial R}{\partial\omega_2}+\frac{\partial R}{\partial\omega_3}=-\frac{\partial R}{\partial\varpi'}. \tag{20}$$

On inserting the values of the derivatives on the left from the canonical equations and integrating, we get

$$\left(1-\frac{i_0'}{i_0}\right)c_1+c_2+c_3=\text{constant}-\int\frac{\partial R}{\partial\varpi'}dt, \tag{21}$$

or, by means of equations (2) and (6), we have

$$(1+z)^{-\frac{1}{3}}\left(1-\frac{i_0'}{i_0}-\tfrac{1}{2}e_1^2-\Gamma_1\right)=\text{constant}-\int\frac{1}{c_0}\frac{\partial R}{\partial\varpi'}dt. \tag{22}$$

After expansion in powers of z and rearrangement, this equation gives

$$e_1^2+2\Gamma_1=\text{constant}+\tfrac{2}{3}z\left(\frac{i_0'}{i_0}-1\right)+\int\frac{2}{c_0}\frac{\partial R}{\partial\varpi'}dt+\text{higher powers of } z \\ +\text{products of } z \text{ with } (e_1^2+2\Gamma_1). \tag{23}$$

When $e'=0$ and we neglect all but the lowest powers of z, e_1^2, $2\Gamma_1$, the equation can be written

$$e_1^2-e_0^2+2(\Gamma_1-\Gamma_0)=\tfrac{2}{3}(z-z_0)\left(\frac{i_0'}{i_0}-1\right), \tag{24}$$

where e_0, Γ_0, z_0 are the values of e, Γ, z at some particular instant.

§ 5. *The Case* $e'=\Gamma=0$.—The arguments present in the development of R have the form $i(\omega-\omega')\pm j(\omega-\varpi)$, where i, j are positive integers including zero. The arguments remaining, after the short-period terms have been eliminated, are given by $i\pm j=pi_0$, $i=pi_0'$, where i_0, i_0' have no common divisor, and p is a positive integer. Hence with the definition of τ in § 2 the former arguments have the form $p[i_0\tau-(i_0-i_0')\varpi]$. According to the

definition in § 1, there is only a single pair of values of i_0, i_0' for a given resonance case, and therefore all the angles present are multiples of the angle

$$N = i_0\tau - (i_0 - i_0')\varpi. \tag{25}$$

The coefficients of those terms have, as a factor, e raised to the power $p(i_0 - i_0')$, so that the principal term is that for which $p = 1$. Hence

$$R_c = A_0 + \sum A_p \cos pN, \qquad (p = 1, 2 \ldots) \tag{26}$$

where A_0, A_p are functions of α, e^2 only, *i.e.* of c_1, c_2, or of z, e_1^2 since we are neglecting the inclination.

In order to obtain a first approximation to the solution of the equations for z, τ, we shall confine our work to the lowest powers of $m^{\frac{1}{2}}$, e present in any equation. Thus R_c is reduced to

$$R_c = A_0 + A_1 \cos N, \tag{27}$$

and equations (9) and (13) become

$$\frac{1}{n_0}\frac{d\tau}{dt} = z, \qquad \frac{dz}{dt} = 3\frac{i_0}{c_0}A_1 \sin N. \tag{28}$$

Equations (8) and (24) become

$$z^2 = C - 6\frac{A_0}{n_0 c_0} - 6\frac{A_1}{n_0 c_0}\cos N; \tag{29}$$

$$e_1^2 = \tfrac{2}{3}\left(\frac{i_0'}{i_0} - 1\right)(z - z_0) + e_0^2. \tag{30}$$

The elimination of N between equations (28) and (29) gives

$$\frac{dz}{dt} = \pm\left[\left(3A_1\frac{n_0}{c_0}\right)^2 - \frac{n_0^4}{4}\left(C - \frac{6A_0}{n_0 c_0} - z^2\right)^2\right]^{\frac{1}{2}}, \tag{31}$$

and from the right-hand member of this we can eliminate e_1^2 by means of equation (30). Thus the equation takes the form

$$\frac{dz}{dt} = [f(z)]^{\frac{1}{2}}, \tag{32}$$

the integral of which gives t as a function of z and thence z as a function of t. From the result, τ and the other variables may be obtained. The character of the motion is determined mainly by a discussion of the equation for z.

§ 6. *Character of the Equation for* z.—Consider first A_0 which has the form $A_0 = m(\alpha_0 + \alpha_1 e_1^2 + \alpha_2 e_1^4 + \ldots)/a'$, where the α_i are functions of a/a' and can, in accordance with § 5, be considered as constants. By the use of equation (30) this can be expressed as a power series in z. Hence we have

$$z^2 - C - 6\frac{A_0}{n_0 c_0} = z^2 - C + 6m(\beta_0 + \beta_1 z + \beta_2 z^2 + \ldots).$$

In accordance with the approximation limitations, the terms $\beta_0 + \beta_1 z$ need

alone be retained. The substitution of z_1 for $z+3m\beta_1$, and a suitable change in the arbitrary constant C, will reduce this expression to the form $z_1^2 - C$. This change will affect no part of the argument which follows except that which is concerned with the vanishing of z, which will be changed to the vanishing of z_1. With this understanding we shall drop the term $6A_0/n_0c_0$.

Next put $A_1 = e^\lambda K$, $\lambda = |i_0' - i_0|$, where K is a constant. With this substitution and by means of equation (30) the equation (31) can be written

$$\frac{4}{n_0^2}\left(\frac{dz}{dt}\right)^2 = \left(\frac{6K}{n_0c_0}\right)^2\left[E + \tfrac{2}{3}z\left(\frac{i_0'}{i_0} - 1\right)\right]^\lambda - (z^2 - C)^2,$$

where

$$E = e_0^2 - \tfrac{2}{3}\left(\frac{i_0'}{i_0} - 1\right)z_0, \qquad e_1^2 = E + \tfrac{2}{3}\left(\frac{i_0'}{i_0} - 1\right)z.$$

This equation gives rise to an elliptic integral as long as $\lambda \leqslant 4$. As i_0, i_0' are to be relatively prime, the pairs of values of i_0, i_0' when $i_0' \leqslant i_0$ thus included are : 1, 1 ; 1, 2 ; 1, 3 ; 1, 4 ; 1, 5 ; 2, 5 ; 2, 3 ; 3, 4 ; 3, 5 ; 3, 7 ; 4, 5 ; 4, 7. All the cases of special interest in the solar system, where three bodies only are involved, are presented in this list.

§ 7. *Particular Cases of Resonance.*—The first case is that in which $i_0 = i_0' = 1$. This is the case of the Trojan group of asteroids. The next group, in which i_0/i_0' has the values 1/2, 2/3, 3/4 . . ., gives $\lambda = 1$, so that the first power of the eccentricity is a factor of the principal term. For these ratios the mean values of a/a' are 0·64, 0·76, 0·82 . . . respectively.

The group 1/3, 3/5, 5/7 . . . has the square of the eccentricity as a factor. To discuss this group properly the inclination should be included, since it gives terms of the same order when the inclination and eccentricity have the same order of magnitude. Thus for the ratio 1 : 3, when $e' = 0$, there are two principal librating angles, $\omega_1 - 3\omega_1' + 2\varpi$ and $\omega_1 - 3\omega_1' + 2\theta$.

§ 8. It is my purpose here to discuss the case 1/3. As mentioned above we shall have two librating angles, since we must include the inclination. We shall discuss each angle separately. To do this we shall take (*a*) $\Gamma = 0$, (*b*) $e = 0$.

§ 9. *The Case 1/3 with* $\Gamma = 0$. *Change of Scale.*—For this case we have $i_0 = 1$, $i_0' = 3$. We shall denote the principal librating term in R by $e_1^2 K \cos N$. The results in § 5 give the following equations to determine the five unknowns, τ, z, N, ϖ, e :—

$$\frac{1}{n_0}\frac{d\tau}{dt} = z, \tag{33}$$

$$\frac{dz}{dt} = \frac{3}{c_0}e_1^2 K \sin N, \tag{34}$$

$$N = \tau + 2\varpi, \tag{35}$$

$$e_1^2 = E + \tfrac{4}{3}z, \tag{36}$$

$$z^2 = C - \frac{6he_1^2K}{n_0c_0} - \frac{6e_1^2K}{n_0c_0}\cos N. \tag{37}$$

Let us now make a change of scale. Write ωz, $1/n_0\omega \cdot t$, $(\omega/3)^{\frac{1}{2}} \cdot e$, for z, t, e_1 respectively, where $\omega = 2K/n_0c_0$. The equations now become

$$\frac{d\tau}{dt} = z, \tag{33'}$$

$$2\frac{dz}{dt} = e^2 \sin N, \tag{34'}$$

$$N = \tau + 2\varpi, \tag{35'}$$

$$e^2 = E + 4z, \tag{36'}$$

$$z^2 = C - he^2 - e^2 \cos N. \tag{37'}$$

We are taking throughout this discussion $e' = 0$. Substituting the value of e^2 from equation (36') in the second term of the right-hand member of equation (37'), we see that it takes the form

$$z_1^2 = C - e^2 \cos N, \tag{38}$$

where $z_1 = z +$ a constant, and C is now a different constant. Substituting z in terms of z_1 in the above equations we see that they retain the same form, provided we make appropriate changes in the constants. With this understanding we shall drop the suffix, and equation (38) will now become

$$z^2 = C - e^2 \cos N. \tag{39}$$

Combining this with equation (34') to eliminate N, we obtain

$$4\left(\frac{dz}{dt}\right)^2 = (E + 4z)^2 - (z^2 - C)^2. \tag{40}$$

§ 10. *Conditions to be satisfied.*—We have the following four conditions :—

(1) Since z measures the deviation of a it must be an oscillating function, and must therefore lie between finite limits which will be denoted by $s \pm d$. We shall take d to be a positive quantity, so that whatever be the size of s the upper and lower limits of z are respectively $s + d$ and $s - d$.

(2) It follows then that the limiting values of z are given by $dz/dt = 0$, so that they are the roots of

$$(E + 4z)^2 - (z^2 - C)^2 = 0. \tag{41}$$

(3) The first form of libration is that in which z can take the value zero. The limiting cases between libration and non-libration are therefore given by $s = d$ when s is positive, and by $s = -d$ when s is negative.

(4) The second form of libration is that in which the angle N cannot take all values but oscillates between limits.

§ 11. By making use of the fact that $s \pm d$ are two roots of equation (41) we get

$$(E + 4s + 4d)^2 = [(s + d)^2 - C]^2,$$
$$(E + 4s - 4d)^2 = [(s - d)^2 - C]^2.$$

From these two equations we find that the values of C and E in terms of s and d are

$$C = s^2 + d^2 - 4d, \qquad E = 2s(d-2).$$

Denoting the roots of equation (41) by z_1, z_2, z_3, z_4, we have

$$\begin{aligned} z_1 &= 2 + (4 + C + E)^{\frac{1}{2}} = s + d, \\ z_2 &= 2 - (4 + C + E)^{\frac{1}{2}} = 4 - s - d, \\ z_3 &= -2 + (4 + C - E)^{\frac{1}{2}} = s - d, \\ z_4 &= -2 - (4 + C - E)^{\frac{1}{2}} = -4 - s + d. \end{aligned}$$

The equation (40) can now be written

$$-4\left(\frac{dz}{dt}\right)^2 = [(z-s)^2 - d^2][(z+s)^2 - (4-d)^2].$$

Also we have

$$e^2 = E + 4z = 2s(d-2) + 4z.$$

Therefore the maximum and minimum values of e^2 are given by

$$e^2 = 2d(s \pm 2).$$

§ 12. *The Libration of* z.—Since z by hypothesis is to lie between $s+d$ and $s-d$, we must have the following conditions satisfied: either (*a*) $s+d > s-d > 4-s-d$ or $-4-s+d$; or (*b*) $s-d < s+d < 4-s-d$ or $-4-s+d$. The former condition gives $s > 2$ and $s+2 > d$. The latter condition gives $s < -2$ and $s+d < 2$. This latter must obviously be discarded since it would make e^2 negative. We therefore can have no libration of z for negative values of s. Thus z can only librate in the region fixed by condition (*a*). Fig. 1 illustrates this libration of z. It is to be remembered that the limiting case between libration and non-libration of z is given by $s = d$, when s is positive.

§ 13. *Motion of a Simple Pendulum.*—Before taking up the libration of the angle N we shall discuss the motion of a simple pendulum which can oscillate or make complete revolutions in a vertical plane. As suggested by Professor Brown this has a close analogy with the resonance problem and is helpful towards an understanding of it.

If l be its length and θ the angle which it makes with the downward vertical at time t, its equation of motion is

$$l\frac{d^2\theta}{dt^2} + g \sin\theta = 0,$$

from which the first integral is obtained,

$$\frac{d\theta}{dt} = \pm\left(C + \frac{2g}{l}\cos\theta\right)^{\frac{1}{2}}.$$

If $C > 2g/l$, $d\theta/dt$ never vanishes, and the solution has the form

$$\theta = \lambda t + \mu + \sum K_i \sin i(\lambda t + \mu),$$

where μ is an arbitrary constant and λ, K_i depend on the arbitrary constant C. The pendulum makes complete revolutions in either sense with a velocity which is a maximum when $\theta = 0$ and a minimum when $\theta = \pi$. If $C < 2g/l$ we can put $Cl = -2g \cos \alpha$, and θ then oscillates between $\pm \alpha$, the solution being of the form

$$\theta = \sum K_i' \sin i(\lambda' t + \mu'),$$

where μ' is an arbitrary constant and K_i', λ_i', depend on C. When, however, $C + 2g/l$ is very small, λ' is nearly equal to $(g/l)^{\frac{1}{2}}$. In the limiting case $C = 2g/l$, and we get

$$\frac{d\theta}{\cos \frac{1}{2}\theta} = \pm 2\left(\frac{g}{l}\right)^{\frac{1}{2}} dt, \qquad \tan \tfrac{1}{4}(\theta + \pi) = A \exp \pm \left(\frac{g}{l}\right)^{\frac{1}{2}} t,$$

where A is arbitrary. When $\theta = \pi$ we have $t = \pm \infty$, and the pendulum takes an infinite time to reach or leave its highest position where both velocity and acceleration are zero. A physical interpretation is to regard the time spent at the highest point as indeterminate, since a minute impulse, too small for our capacity to measure, will alter this by a finite amount. Further $d\theta/dt$ has no mean value independent of t. Since the motion must be real, $C > -2g/l$.

§ 14. *The Libration of* N.—By means of equation (36′) we have

$$z^2 = C - e^2 \cos N = C - (E + 4z) \cos N. \tag{42}$$

Differentiating with respect to the time, we obtain

$$2z\frac{dz}{dt} = \frac{dN}{dt}(E + 4z) \sin N - 4\frac{dz}{dt} \cos N. \tag{43}$$

But equation (34′) gives

$$2\frac{dz}{dt} = e^2 \sin N = (E + 4z) \sin N, \tag{44}$$

and substituting this value for dz/dt in (43) we get

$$\frac{dN}{dt} = z + 2 \cos N. \tag{45}$$

From the analogy of the simple pendulum it is seen that the condition for N to librate is that its derivative with respect to the time shall vanish. Thus for the angle N to librate we have

$$z + 2 \cos N = 0. \tag{46}$$

Eliminating N from this equation by means of equation (42) we have

$$2z^2 + Ez + 2C = 0. \tag{47}$$

For z to be real it is seen that it is necessary that $E^2 > 16C$. Let us now translate this condition into terms of the variables s and d. From § 11 we found that $C = s^2 + d^2 - 4d$ and $E = 2s(d - 2)$. By means of these relationships the condition becomes $(s^2 - 4)(d - 4) > 0$. We now have two possibilities: either (*a*) $d > 4$ and $s^2 > 4$, or (*b*) $d < 4$ and $s^2 < 4$. And now from these facts

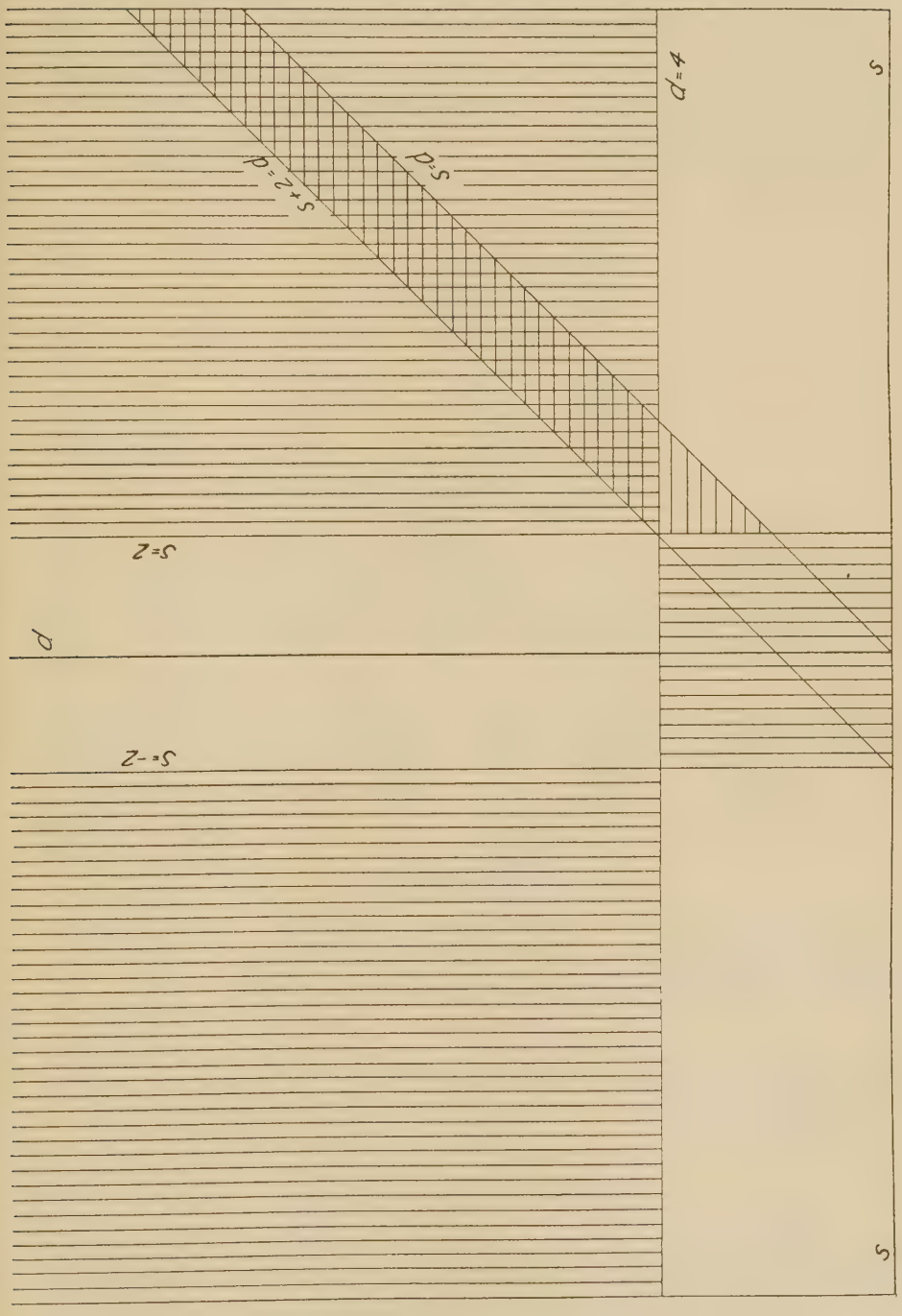


FIG. 1.—*Horizontal shading indicates region of librating z ; vertical shading, librating* N. *Case* I′ = 0.

we are able to locate the regions in which the angle N can librate. In any other regions but these for N to librate we would get an imaginary value for z which is of course impossible. The librating regions for N are illustrated in fig. 1.

§ 15. *The Case* 1/3 *with* e = 0. *Change of Scale.*—We once more have $i_0 = 1$ and $i_0' = 3$. We shall denote the principal librating term by $\Gamma K \cos N$ in the disturbing function R.

The results in § 6 give us the following equations to determine the five unknowns, τ, z, N, θ, Γ :—

$$\frac{1}{n_0}\frac{d\tau}{dt} = z, \tag{48}$$

$$\frac{dz}{dt} = \frac{3}{c_0}\Gamma K \sin N, \tag{49}$$

$$N = \tau + 2\theta, \tag{50}$$

$$\Gamma = E + \tfrac{2}{3}z, \tag{51}$$

$$z^2 = C - \frac{6hK\Gamma}{n_0 c_0} - \frac{6\Gamma K}{n_0 c_0}\cos N. \tag{52}$$

Making a change of scale, we shall put ωz, $1/n_0\omega \,.\, t$, $\omega/3 \,.\, \Gamma$ for z, t and Γ respectively, where $\omega = 2K/n_0 c_0$. The above equations now become

$$\frac{d\tau}{dt} = z, \tag{48'}$$

$$2\frac{dz}{dt} = \Gamma \sin N, \tag{49'}$$

$$N = \tau + 2\theta, \tag{50'}$$

$$\Gamma = E + 2z, \tag{51'}$$

$$z^2 = C - h\Gamma - \Gamma \cos N. \tag{52'}$$

Throughout this discussion we once more take $e' = 0$. Substituting the value of Γ given by equation (51′) in the second term of the right-hand member of equation (52′), we see that it will take the form

$$z^2 = C - \Gamma \cos N, \tag{53}$$

where appropriate changes have been made in the constants. If we now combine this equation with equation (49′) in order to eliminate the angle N we obtain the result,

$$(E + 2z)^2 - (z^2 - C)^2 = 4\left(\frac{dz}{dt}\right)^2. \tag{54}$$

Once more we shall take the upper and lower limits of z to be $s + d$ and $s - d$ respectively, where s and d are defined as before. We therefore have that the extreme values of z are given by the equation

$$(E + 2z)^2 = (z^2 - C)^2. \tag{55}$$

Using the fact that $s+d$ and $s-d$ are roots of this equation, we get

$$(E+2s+2d)^2=(s^2+2sd+d^2-C)^2,$$
$$(E+2s-2d)^2=(s^2-2sd+d^2-C)^2.$$

Solving these for E and C, we have

$$E=2s(d-1), \qquad C=s^2+d^2-2d.$$

The roots of equation (55) are now

$$z_1= \quad 1+(1+C+E)^{\frac{1}{2}}=s+d,$$
$$z_2= \quad 1-(1+C+E)^{\frac{1}{2}}=2-s-d,$$
$$z_3=-1+(1+C-E)^{\frac{1}{2}}=s-d,$$
$$z_4=-1-(1+C-E)^{\frac{1}{2}}=-2-s+d.$$

And we can write the equation (54) as follows:—

$$4\left(\frac{dz}{dt}\right)^2=[(z-s)^2-d^2][(z+s)^2-(d-2)^2].$$

We also have

$$\Gamma=E+2z=2s(d-1)+2z,$$

and therefore the maximum and minimum values of Γ are given by

$$\Gamma=2d(s\pm 1).$$

§ 16. *The Libration of* z.—Since z by hypothesis is to lie between $s+d$ and $s-d$, we must have the following conditions satisfied: either (a) $s+d>s-d>2-s-d$ or $-2-s+d$; or (b) $s-d<s+d<2-s-d$ or $-2-s+d$. The former gives $s>1$ and $s+1>d$. The latter gives $s<-1$ and $s+d<1$, and this must be discarded since it would make Γ negative. Therefore in this case also we can have no libration of z for negative values of s. Thus z can librate only in the region fixed by the first condition. This is illustrated in fig. 2, where it is seen that z can librate only in the region between the lines $s=d$ and $s+1=d$, where $s>1$.

§ 17. *The Libration of* N.—Equation (51′) gives

$$z^2=C-\Gamma\cos N,$$
$$=C-(E+2z)\cos N. \tag{56}$$

Differentiating this with respect to the time and eliminating dz/dt by means of equation (49′), we obtain

$$\frac{dN}{dt}=z+\cos N. \tag{57}$$

For the angle N to librate, we must have

$$z+\cos N=0. \tag{58}$$

Eliminating N from this equation by means of equation (56), we have

$$z^2+Ez+C=0. \tag{59}$$

The condition for z to be real gives $E^2-4C>0$. Translating this into terms of s and d, we find it gives $(s^2-1)(d-2)>0$; where from § 15 we

have $E=2s(d-1)$ and $C=s^2+d^2-2d$. We now have two possibilities: either (*a*) $s^2>1$ and $d>2$; or (*b*) $s^2<1$ and $d<2$. These fix the regions in which the angle N can librate. They are illustrated in fig. 2.

§ 18. *The Eccentricity.*—We found the maximum value of e^2 to be given by $e^2=2d(s+2)$. The least value of this expression, where z librates, is given by $d=s=2$. This gives $e^2=16$. From § 9 we see that we must multiply this expression by $\omega/3$, where $\omega=2K/n_0c_0$, in order to obtain the value of the square of the eccentricity. From the equation $n^2a^3=n'^2a'^3$, we obtain $a/a'=\alpha=0{\cdot}48$; since $n=3n'$ and $a'=5{\cdot}2$, the mean distance of Jupiter. We obtain c_0 from the equation $1/c_0{}^3=3n'$. This gives $c_0=1{\cdot}58$. Also $n_0=1/c_0{}^3$, hence $n_0=0{\cdot}25$. The eccentricity and the inclination are in general about 1/10 and the mass about 1/1000. For the coefficient K we have (see Tisserand, *Mécanique Céleste*, p. 309)

$$K=\frac{m}{8}(21A^{(3)}+10A_1{}^{(3)}+2A_2{}^{(3)}).$$

On calculation, taking into consideration only the terms up to and including α^3 in the A's, we get $K=0{\cdot}09m$, where m is the disturbing mass. We therefore have

$$e_1{}^2=\frac{32K}{3n_0c_0}=0{\cdot}0025.$$

And we obtain for the least maximum of the eccentricity a value 0·05.

§ 19. *The Inclination.*—The maximum value of Γ is given by $\Gamma=2d(s+1)$. The least value of this expression, when z librates, is given by $d=s=1$. This gives $\Gamma=4$. We must multiply this by $2K/3n_0c_0$, where n_0, c_0 have the same values as before and K is given by

$$K=\frac{mB^{(2)}}{4}=\frac{15m}{16}\frac{a^3}{a'}(1+\tfrac{7}{4}\alpha^2+\ldots).$$

Evaluating only the first term we get for K a value $0{\cdot}02m$. Hence we have

$$\Gamma=\frac{8K}{3n_0c_0}=(0{\cdot}012)^2,$$

which is the least maximum value of Γ. This gives for the inclination a value of about one degree.

§ 20. *The Equation for* N.—For the case 1/3, where Γ is equal to zero, we have $N=\tau+2\varpi$. Differentiating this with respect to the time, we get

$$\frac{dN}{dt}=\frac{d\tau}{dt}+2\frac{d\varpi}{dt}=n_0z-\frac{\partial R}{\partial c_1}-2\frac{\partial R}{\partial c_2}, \tag{60}$$

and differentiating again with respect to t, we have

$$\frac{d^2N}{dt^2}=n_0\frac{dz}{dt}-\left[\frac{\partial^2R}{\partial c_1{}^2}\cdot\frac{dc_1}{dt}+\frac{\partial^2R}{\partial c_1\partial c_2}\frac{dc_2}{dt}+\frac{\partial^2R}{\partial c_1\partial\tau}\frac{d\tau}{dt}+\frac{\partial^2R}{\partial c_1\partial\varpi}\frac{d\varpi}{dt}\right]$$
$$-2\left[\frac{\partial^2R}{\partial c_1\partial c_2}\frac{dc_1}{dt}+\frac{\partial^2R}{\partial c_2{}^2}\frac{dc_2}{dt}+\frac{\partial^2R}{\partial c_2\partial\tau}\frac{d\tau}{dt}+\frac{\partial^2R}{\partial c_2\partial\varpi}\frac{d\varpi}{dt}\right]. \tag{61}$$

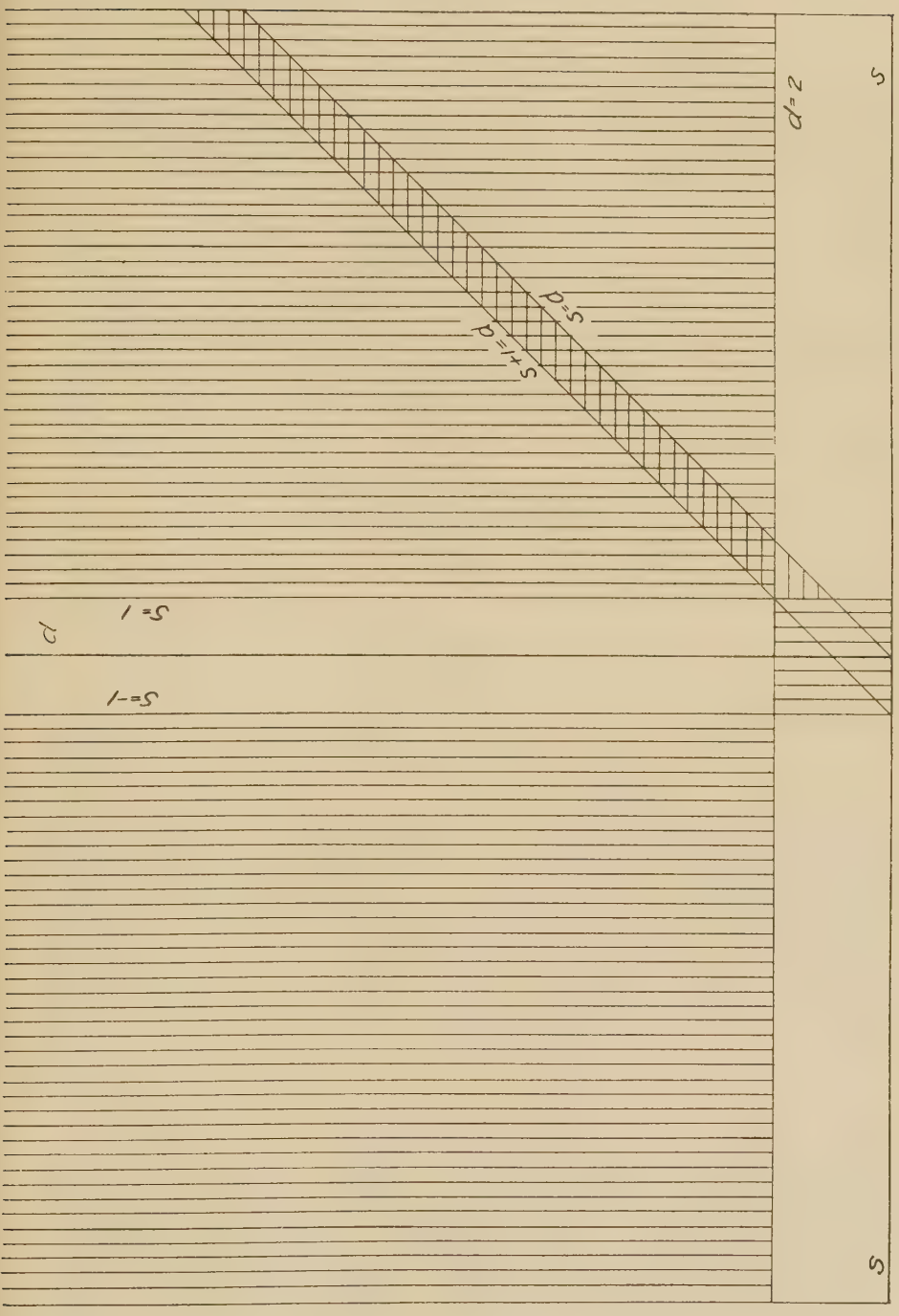


FIG. 2.—*Horizontal shading indicates region of librating z ; vertical shading, librating* N. *Case* e = 0.

If we neglect all terms factored by the square of the disturbing mass, we have

$$\frac{d^2N}{dt^2} = n_0\frac{dz}{dt} - \frac{\partial\tau}{\partial t}\left(\frac{\partial^2 R}{\partial c_1 \partial\tau} + 2\frac{\partial^2 R}{\partial c_2 \partial\tau}\right). \tag{62}$$

We can neglect the term $\partial^2R/\partial c_1\partial\tau$, since with respect to the term $\partial^2R/\partial c_2\partial\tau$ it is factored by e_1^2. The term n_0dz/dt is of order me_1^2, and the second term of the right-hand member is of order mz. We have, then,

$$\frac{d^2N}{dt} = n_0\frac{dz}{dt} - 2\frac{\partial\tau}{\partial t}\frac{\partial^2 R}{\partial c_2\partial\tau}. \tag{63}$$

If we now make use of the equations of § 9 before we made the change of scale, we get

$$\frac{d^2N}{dt^2} = n_0\frac{dz}{dt} + 2n_0 z\frac{\partial(Ke_1^2)}{\partial c_2}\sin N, \tag{64}$$

from which we have

$$\frac{d^2N}{dt^2} = \frac{3n_0e_1^2K}{c_0}\sin N - \frac{4n_0zK}{c_0}\sin N, \tag{65}$$

and from equation (36) we obtain

$$\frac{d^2N}{dt^2} = \frac{3En_0K}{c_0}\sin N. \tag{66}$$

This can be written

$$\frac{d^2N}{dt^2} + \frac{3En_0K}{c_0}\sin(N \pm \pi) = 0. \tag{67}$$

This is the fundamental equation for the libration of N. It is the ordinary equation of the simple pendulum. By § 11 we see that N takes the value $+\pi$ at the upper limit of z, and hence we can have small oscillations in the neighbourhood of $z = s + d$. The value of the coefficient of $\sin(N \pm \pi)$ is approximately 5×10^{-7}. When $\Gamma = 0$ the general value of N is $N = i_0\tau - (i_0 - i_0')\varpi$. Here the eccentricity enters into the power $(i_0' - i_0)$. We obtain for the general equation for the libration of N,

$$\frac{d^2N}{dt^2} = \frac{n_0}{2c_0}Ke_1^{i_0'-i_0-2}[6i_0e_1^2 - 3i_0(i_0' - i_0)(e^2 - E)]\sin N.$$

We see that the coefficient of $\sin N$ is of order $e^{i_0'-i_0-2}$ times the disturbing mass. For the case 1/3 we have $i_0 = 1$ and $i_0' = 3$. The fundamental equation for the libration of $N = \tau + 2\theta$, where we take into consideration the inclination but neglect the eccentricity, has the same form as the above except that E and K have different values.

Yale University :
1932 February.

THE SELENOLOGICAL ASPECT OF THE LUNAR SURFACE.

H. G. Tomkins, C.I.E., F.R.P.S. (Plates 7, 8, 9.)

1. No one who has systematically observed the lunar surface can have failed to notice that the formations he sees were not all made at the same time, or even within long periods of each other. Craters like Copernicus, Tycho, Theophilus, Aristarchus and others must obviously have been formed long after the plains or other surfaces on which they stand ; the "flows" which have been referred to in previous papers * must certainly have been active ages later than the *maria* over which they pass ; the Nectaris "flow," which has probably seen about five revolutions in its history, must have cut through the Hæmus Mountains and the Apennines long after these mountains were formed.

Similarly, in the case of many other formations it is probable that the changes, of which they afford evidence, occupied hundreds of centuries and more, and on the Earth it is now known that prodigious periods of time elapsed during the building of the surface features, and such changes are still going on.

So it seems to have been, and possibly may still be, with the lunar surface, for in Nature, as far as is known, nothing stands still.

2. In the case of the Earth the history of its surface has been divided into *ages*, and the question arises whether some such method cannot also be adopted in the study of the Moon.

The *relative* ages of many of the lunar formations can be determined by their comparison with one another and with the surface round them, but a difficulty arises in converting these into a uniform classification. For instance, it can easily be shown that Aristarchus is a younger formation than Kepler, and Kepler than Copernicus, and so on ; but the series thus obtained do not always connect, and the need arises for a basis from which to classify them as a whole.

3. In the case of the Earth such a basis has been supplied by an *assumed* datum surface known by geologists as the "substratum."

A particularly clear description of this is given by Daly in his book, *Our Mobile Earth*, as follows :—

"Basalt dominates among the lavas now being erupted on the continents as well as on the floors of the oceans. This and other facts familiar to geologists tend to show that the crystallized basaltic earth shell below each ocean basin and, as well, each of the high floating continents are underlain by a deep continuous layer of non-crystallized, glassy, eruptible basalt. This layer is a complete earth shell which at a given level is everywhere almost perfectly uniform in composition, though the shell grows slightly denser with each 10-mile increase in depth. For convenience this potentially

* *M.N.*, **90**, 198, 767, 1929.

eruptible layer of basalt will be called the 'substratum,' as it is a world-encircling shell underlying the manifestly crystallized, non-eruptible skin of the globe.

"The Earth's skin of strong solid rocks is fittingly described by the old popular name, 'crust'."

This assumed "substratum" referred to by Daly for the Earth would probably be some 15 km. below the continents, but geologists are handicapped by lack of data regarding the composition of sub-oceanic rocks.

4. It is proposed in this paper to examine the case for a "substratum" in the case of the Moon, with a view to building up a selenological record ; at the outset, however, observation indicates that such a substratum would probably be situated at no distance below the surface which would be comparable with that assumed in the case of the Earth.

5. The portions of the lunar surface which first merit attention are obviously the *maria,* and in dealing with them there is at this stage one matter only which it is necessary to stress ; namely, the dark patches or areas that appear on, or through, their floors.

The Mare Imbrium (Plate 7) is 340,000 square miles in area, and over the whole of this a black substance is visible, apparently shrouded in some parts by a light transparent covering, but laid bare in others. The most prominent of these latter patches is one south of the Sinus Iridum, where it occupies about a quarter of the *mare* and is of most irregular shape. The photographs suggest that to the east of the Sinus Iridum this black substance underlies the peninsula on which the crater Mairan stands, and issues again as a dark patch on the southern side of the Sinus Roris. Similarly, in the Sinus Iridum the dark substance seems to proceed under the "land" and appear again at the west end of the Sinus Roris.

A smaller example of a similar thing is a patch between the Straight Range and the Teneriffe Mountains. The photographs suggest that here the dark material is continuous under a thin coat of a lighter material, and issues in two small, but distinct, patches on its northern side.

Again, another example occurs on the floor of Plato, the western three-quarters of which is very dark, and a small lighter patch of some kind remains —indeed, all down the strip of land on which Plato stands, between the Mare Imbrium and the Mare Frigoris and Sinus Roris, it seems probable that the dark, and photographically black, material is not far below the surface. On the west side of the Mare Imbrium it likewise looks as if the dark underlying matter were only partially obscured by the white covering.*

An extension of the large dark patch already mentioned is also apparent between the two long northern rays from Copernicus, and extends also up the western side of that formation, where it stands out sharply in three places. Similarly also on the south and east of the formation the dark area is visible, and the conclusion appears to be unavoidable that Copernicus has been erupted through it and that the white rays from the crater lie on its surface.

6. It remains to draw attention to the region round Archimedes, just east of which crater a further dark area appears. This again is evidently part

* These dark areas should not be confused with the " flows."

of the same dark substance as before, its shape having been determined by the later features upon it.

The whole of the portion of the *mare* on which Archimedes, Autolycus and Aristillus stand has been more or less masked by the white rays from the two latter formations, which are of course more recent than the craters and, consequently, the *mare*.

7. It may perhaps be mentioned here that the white rays across the floor of Archimedes originate in the crater Autolycus and form part of that system. This has been already noticed by others, but on the photographs taken by the present writer these rays are also shown to pass into Archimedes through two gaps in the western wall of that crater, across the floor, and then out through two other gaps in the eastern wall, and across the floor of the Mare Imbrium nearly as far as the first long white ray from Copernicus, *i.e.* about half-way across the Mare Imbrium. Consequently they evidently belong to Autolycus and not Archimedes. There does not appear to be any previous record of this portion of the rays over the *mare*.

8. Returning now to the irregular dark area at the east end of the Mairan peninsula, it will be noticed that this extends towards Aristarchus and beyond into the Mare Procellarum, over which further extensive dark patches are visible, and, indeed, so marked are they round Kepler as to give that formation the appearance of being situated on an area raised above the general surface of the *mare*. This elevation would be quite likely to occur if Kepler were upheaved from below in the course of its formation, in the manner suggested by the writer some years ago.* Round the margin of this island further irregular dark markings can be seen, a portion of them extending to and invading the floor of Flamsteed and beyond. Similarly, dark areas occur on the *mare* opposite Grimaldi, and also on the floor of the crater itself, as well as on that of Riccioli. Farther south other contrasts occur, notably in the case of the crater Billy and several spots round it.

9. Proceeding farther south, another dark patch is found just south-west of the Riphæan Mountains. Most of these patches, however, may be upthrusts from below, as, of course, is the case with the "flow" at the southern end of the Mare Imbrium. Omitting these, and proceeding to the Sinus Medii, this formation is probably underlain by a dark substance which is visible through the floor in two places ; and a similar thing occurs at the edge of the Sinus Æstuum. In the Mare Vaporum is a small "flow" to which attention has already been drawn elsewhere.†

10. The Mare Humorum is omitted for the present, as there are reasons for holding that this area is not a "sea" in the sense used for the others dealt with, but that it had a later origin, or at any rate that the dark markings on its floor, like those round it, were more recent.

11. Taking the Mare Serenitatis next, the greater part of this "sea" seems to have been overlain by a very shallow surface covering, through which the dark substratum can be seen, in some places more clearly than in others. One of these is the spot in the neighbourhood of Mount Acherusia, and another less evident is situated in the north-east corner of the *mare*.

* *Journ. B.A.A.*, **18**, 361, 1908. † *M.N.*, **90**, 8, 1930.

There is also a patch just north of the crater Aratus at the foot of the Hæmus Mountains, but this last may possibly be part of the "flow" through that range.

There are some other patches on the floor of the *mare* darker than the general surface, notably a wide curved one on the western part and a straight strip from just east of Bessel down to the north-east patch already mentioned ; but these are possibly due to variation in the surface substance overlying the darker floor.

12. Adjoining the Mare Serenitatis is the Mare Tranquillitatis, an apparently very dark-floored area but one which on careful examination turns out to be about the same shade as the Mare Serenitatis, and which, like that "sea," was covered originally by a surface substance of some kind. Superposed on this is the "flow" from the Mare Nectaris which has traversed the Mare Tranquillitatis, passing over the low parts, round the high parts, and through the Hæmus Mountains and Mare Serenitatis as described in a previous paper.*

The deviations round obstructions such as Jansen, Vitruvius, Dawes and Plinius, to say nothing of its passage through the Hæmus Mountains, are truly remarkable ; and the cause of the comparative darkness of the floor of Mare Tranquillitatis seems to be explained by the fact of this enormous eruptive stream from Mare Nectaris flowing over the already dusky surface of an earlier period.

13. It will of course be noticed that just inside the place of issue from Mare Nectaris, and also beyond it for some distance, the stream is very much lighter in tint till it joins Mare Tranquillitatis. It would seem that the more recent and lighter formations of Theophilus and Mädler and their terrain, which break the surface of ground over those parts of the "flow," account for the difference in tint.

The abrupt stoppage of the "flow" by the Palus Somnii is also noticeable.

14. Coming now to the Mare Fœcunditatis, some difficulties are felt owing to its nearness to the Moon's west limb and the foreshortening thereby caused. Two plates taken on 1929 November 14 and 1931 March 31, the terminator being at about $+65°$ and $+53{\cdot}5°$ respectively, show the "sea" to be apparently a self-contained, heavily erupted area, the features on it of course being of later date—Langrenus markedly so, with its rays extending over the dark floor. North of Langrenus also there is a group of dark patches which are evidently eruptive and similar to others already referred to.

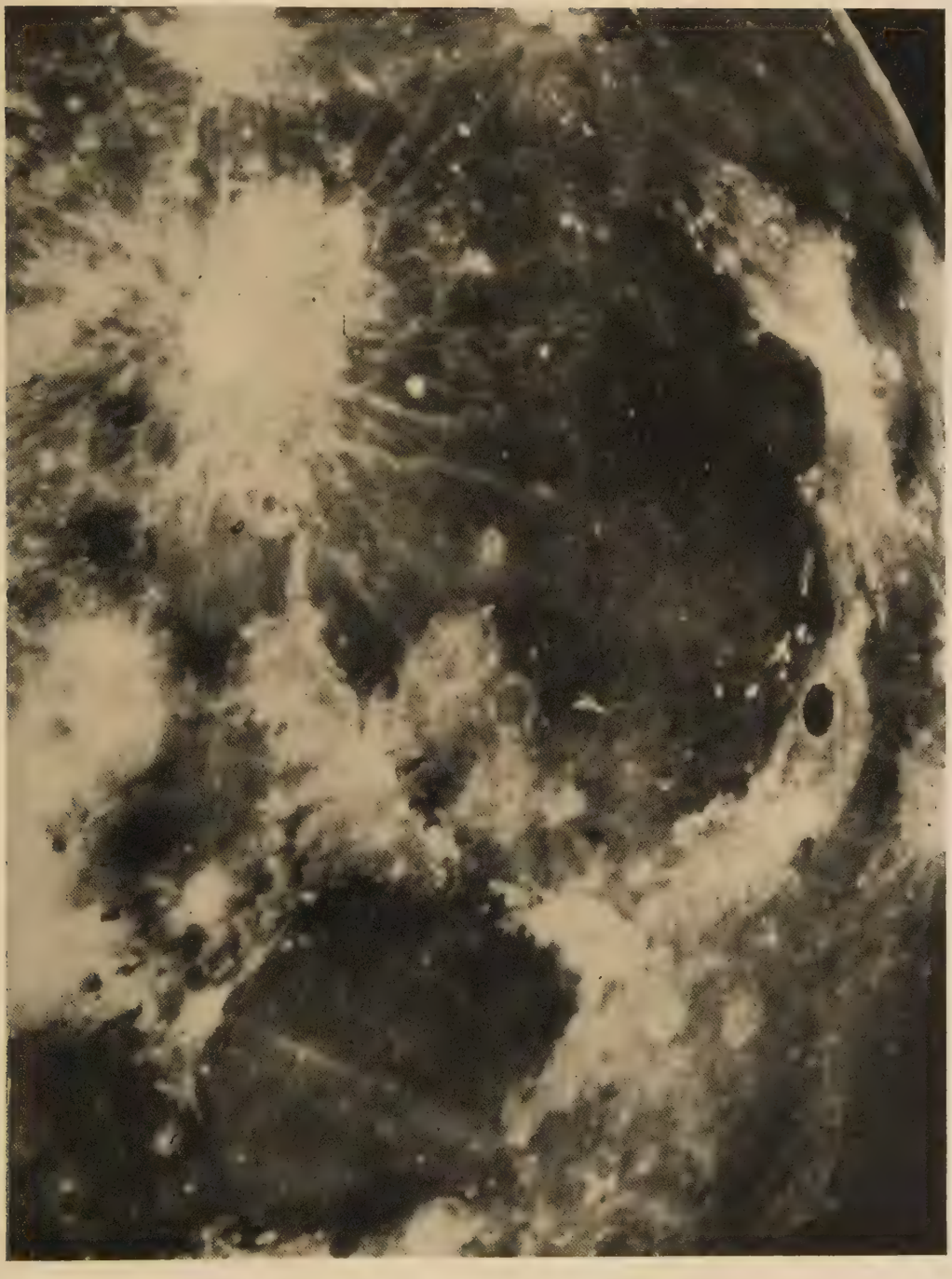
This explanation is also supported by the small eruptive areas all down this limb of the Moon, even appearing on a portion of the Mare Crisium on its western floor and just inside its western wall, through which it breaks out a little farther north on to the plain beyond.

Still farther north, on the limb of the Moon, is another group of eruptive formations, the chief of which is the crater Endymion, but the whole tract is affected.

15. The three "seas," Crisium, Frigoris and Humorum, now remain to be mentioned. They are all three situated very near to the Moon's limb, and consequently are so considerably foreshortened as to be adversely affected

* *M.N.*, **90**, 198, 1929.

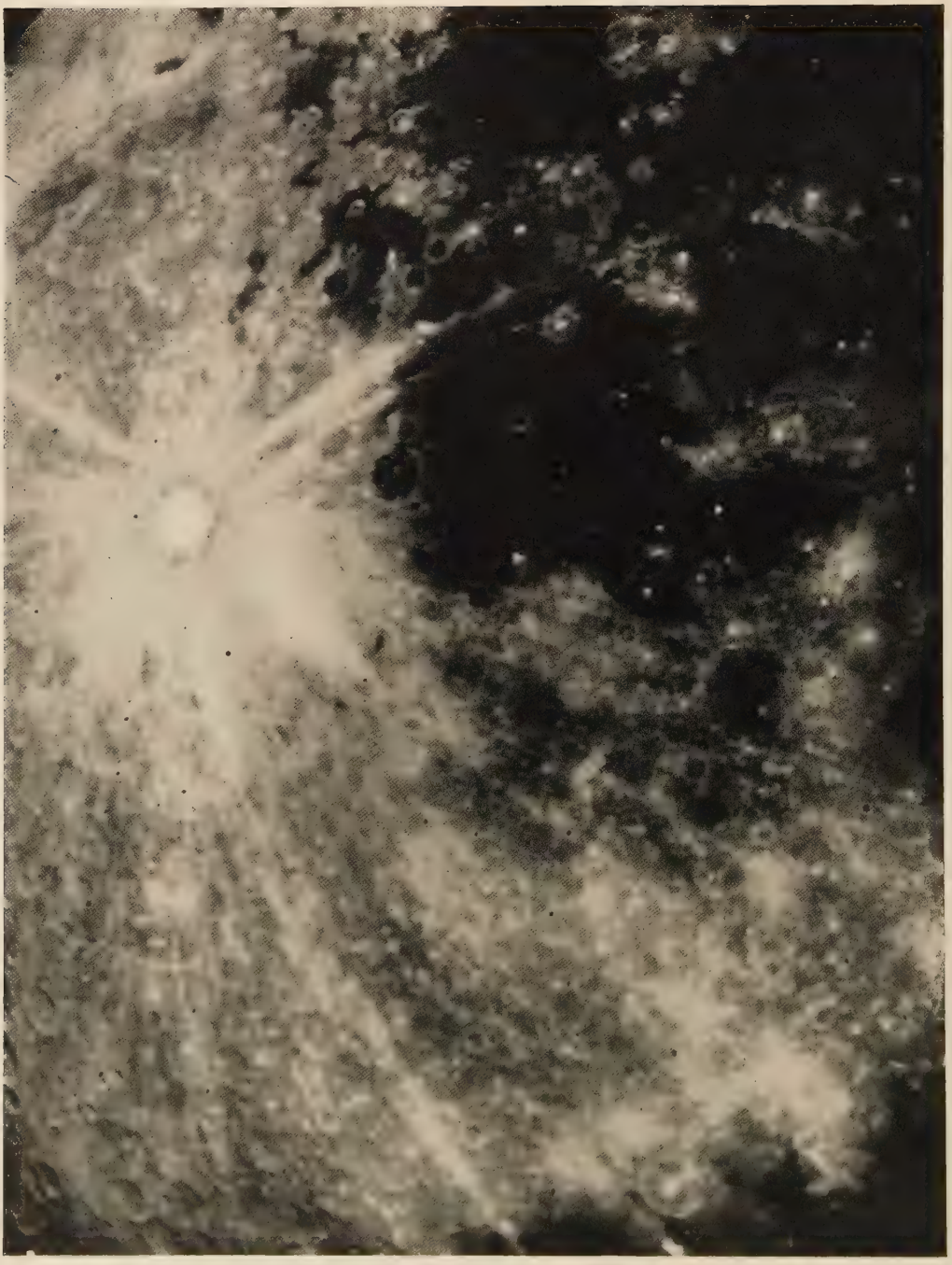
E



W.

Defects on Plate: two on N. and W. of Copernicus.

N



W.

Defect on Plate: one on S.E. of Tycho.

S.

N.

Defects on Plate: two on S. and W. of Mare Crisium.

for use in this paper. The Mare Frigoris is mainly taken up by the Nectaris "flow," as shown in a previous paper.* The Mare Humorum, like the Mare Nectaris, has evidently been the scene of extensive eruptions which have burst through its walls near the crater Gassendi. The Mare Crisium seems to be merely a large formation similar in many ways to other large craters on the Moon.

These formations were probably the scenes of eruptions later in age than the floors of the other *maria*, and cannot therefore be considered as fundamental areas for the purposes of the present investigation.

16. It is now necessary to refer to a peculiar mottled appearance of the land areas which appears on the photographs of the surface. It first came to notice in the course of the present investigation, and was put to a thorough test in order to insure that the appearance was not due to any peculiarities of the plate on which it was first found. On comparison with other plates, however, the details were seen to correspond from plate to plate, and are therefore real. This was also confirmed by a reference to the makers of the plates. It seems to be of far-reaching importance, as the peculiarity is not only prominent round the Mare Imbrium, where it was first noticed that the dark areas on the floor give one a strong impression that they extend under the northern border of the "sea" and appear on the opposite side of the peninsula, but can be seen also on most of the other photographs of the "sea" coasts, as well as practically all over the lunar land surface. The effect is visible on the photographs taken under high illumination; and on Plate 8, taken just after full Moon, it is particularly evident, both on the west and north of Tycho, and even more so on the east.

In this last case an explanation of the matter may probably be found, as over this part of the Moon the surface crust has diminished and disappeared in places, leaving the underlying dark substratum visible. A similar thing has presumably also occurred (Plate 9) in the neighbourhood of the Mare Crisium near the western limb of the Moon.

Incidentally, the above may possibly explain the hitherto puzzling dark ring round the crater Tycho, on the assumption that there is a thinner covering there than elsewhere over the dark material below.

17. As has just been stated, the plates were taken at full or nearly full Moon, when the illumination was vertical or nearly so, and hence shadows are absent. The result therefore is one of pure contrast, and the photographs leave one with the impression that the lunar surface-covering consists of a thin, broken-up layer of material resting on a continuous dark *substratum* that is plainly visible in the *maria*, and only less plainly in the case of the land areas all over the Moon. The effect of this conveyed over the distance to the Earth would probably be an appearance such as is shown in the photographs. The surface-covering in such case may be comparable with pumice, or volcanic ash, as has been suspected from photometric observations made by Schönberg and Brunn,† or may even be in the nature

* *M.N.*, **90**, 198, 1929.

† Schönberg, "Theoretische Photometrie," chap. i of vol. ii, *Handbuch der Astrophysik*; Brunn, *Das Weltall im Bilde.*, II. "Der Mond."

of an efflorescence rather than a crust such as that on the Earth. This seems to be the most promising explanation of the "mottling" of extensive areas all over the lunar surface.

18. As regards the process by which the "contrast" photographs of the full or nearly full Moon have been obtained, it should be mentioned that a new plate known as the "Astra plate," made by Messrs. Ilford, Ltd., was used. It has a process type emulsion giving great density and contrast. The grain is very fine, and the plate is very highly colour-sensitised. Its *H* and *D* speed is about 30, but in lunar work with the Dedham 24-inch Cassegrain reflector, equivalent focal length of 47 feet, the exposure for a lunar photograph with an orange filter, which was used for all the plates reproduced here, is found to be about two seconds. This additional speed is said to be due to density building up very much more rapidly on the new plate than on the panchromatic. The plates were of course backed to prevent halation trouble, and were loaded into the dark slides and handled in the dark room in absolute darkness. After exposure, the backing was wiped off with a damp rag in absolute darkness, and the plate was then dropped into a dish containing red desensitol sufficient to insure the plate being well covered. After remaining there for three minutes, the plate was developed by candle-light in the ordinary way. The candle was placed behind a ground-glass screen some ten or twelve feet away, and the plate was kept away from the light except when necessary to turn momentarily for inspection. The developer used was Hydroquinone maximum contrast. When development was complete, the plate was fixed as usual, and washed for an hour under a tap. The washing removed the dye, and the plate was then dried. Pinholes unfortunately appear on each plate.

19. In conclusion I should like to acknowledge the kind help and encouragement I have received from the Astronomer Royal and Mr. C. R. Davidson in the matter of filters and in many other ways.

THE ROTATIONAL SPACE MOTIONS OF THE STARS.

J. M. Mohr.

(Communicated by Professor W. W. Heinrich)

I. *Introduction*

1. It is the object of this work to examine by accurate computations the consequences of the theory of the rotation of the galaxy, and to try to explain the series of phenomena which accompany the rotation. The result is a confirmation of the Lindblad-Oort theory,* and a satisfactory explanation of a series of questions unanswered until now.

2. This study is based on 1678 stars of spectral types B, A, F, G, K, M,

* *Arkiv Mat., Astr. och Fysik.*, 19 A, Nos. 21, 27, 35 ; 19 B, No. 7 ; 20 A, No. 3, 1925–1928 ; *Astron. Nachrichten*, **236**, 181–188, 1929 ; *M.N.*, **87**, 553–564, 1927 ; **90**, 503–516, 1930 ; *B.A.N.*, **3**, 275–282, 1927 ; **4**, 79–89, 1927 ; **4**, 269–284, 1928.

for which I have computed space-velocities according to the theory of galactic rotation. The equations for relative velocity-components are :

$$\xi = V \cos \alpha \cos \delta - \frac{k}{\pi}(\mu_\alpha \sin \alpha \cos \delta + \mu_\delta \cos \alpha \sin \delta),$$

$$\eta = V \sin \alpha \cos \delta + \frac{k}{\pi}(\mu_\alpha \cos \alpha \cos \delta - \mu_\delta \sin \alpha \sin \delta),$$

$$\zeta = V \sin \delta + \frac{k}{\pi}\mu_\delta \cos \delta,$$

in which ξ, η, ζ are the velocity-components, V the radial velocity, μ_α, μ_δ the proper motions in right ascension and declination, π the parallax, α, δ the right ascension and declination, and $k = 4{\cdot}737$ km./sec. Radial velocities have been taken mostly from the Mount Wilson Catalogue, the rest from Vôute's Catalogue. Parallaxes are trigonometrical and spectroscopic. In accordance with Balanovsky's * remarks, for nearly all stars with parallaxes over 0″·050 the trigonometrical parallaxes were used; for stars with parallaxes under 0″·050 the spectroscopic parallaxes have been employed.

3. Before any examination of stellar motions can be undertaken it is necessary to reduce the relative motions to real motions, which are unaffected by the solar motion. To derive the velocity of each star the formula

$$V = \sqrt{(\xi + X)^2 + (\eta + Y)^2 + (\zeta + Z)^2}$$

is used, in which ξ, η, ζ are the components of the relative velocity of a star, while X, Y, Z are components of the solar motion. The real stellar motions are therefore dependent on the solar motion. Assuming a circular orbit of the Sun round the galactic centre, the value of the galactic longitude of the direction of solar motion is given by the value of the galactic longitude of the rotational centre, increased by 90°.

Connected with the problem of the rotation is theblem pro of the rotational centre. I shall not investigate the possibility or impossibility of such a rotational centre ; the question is, What galactic longitude we should assign to this point? We shall realise very soon the significance of this question. It is known that the longitude of the centre varies according to which stars are used for each calculation. The method of determining the longitude of the centre is clear and simple. It reduces to the determination of the position of the major axis of the velocity-ellipsoid of one or other type

Table I

Type	Gal. Longitude	Number of Stars
B	342°	128
A	350	219
F giants	344	51
F dwarfs	337	202
G giants	365	360
G dwarfs	334	159

* *Bull. de l'Institut Astr. Leningrad*, No. 11, 81–107, 1925.

of stars. My calculations give (see Table I) the values for the centre of rotation for several types of stars.

I have not computed values for the other spectral types. But from determinations of the galactic centre by other authors (Eddington, Charlier, Raymond, Eddington and Hartley, Beljawsky, Smart, Boss, Raymond and Wilson, etc.)* it is known that stars moving in space with low velocities according to the accepted theory of a solar velocity of 20 km./sec. (in random motions) have higher galactic longitude values than stars moving in space with large velocities. Approximate values obtained in the first case are

$$l = 340° - 350°,$$

and in the second, $l \doteq 320°$.

For all stars therefore the mean value is very nearly 340°. Assuming the Sun to move perpendicular to these directions, and assuming the Sun's orbit round the galactic centre to be perfectly circular, we obtain the following directions of solar motion :—

For stars with low velocity . . .	$l = 70° - 80°$
For stars with high velocity . .	$l \doteq 50°$
For all stars	$l \doteq 70°$.

Strömberg,† in his studies of solar motion relative to globular clusters, determines the direction of the solar motion as being $l = 71°$, $b = 10°$. Oort, however, presumes it to be $l = 55°$, $b = 0°$. From the computations obtained it would appear that Strömberg's values are correct for stars with relatively low space-motions (*i.e.* giants), and Oort's values are suitable for those with high space-motions (dwarfs). Now there is the problem of determining which of these values is correct. The value of the galactic longitude of the centre deduced for giants seems to be more exact, as the general tendency is to think that the velocities of giants are more regular than those of dwarfs. On the other hand, the value suitable for dwarfs is supported by Shapley's valuable investigations, and therefore it is very difficult to decide upon one or other of these values without making an accurate numerical analysis of both possibilities. So I was forced to take both values into account and to calculate the stellar velocities for both directions of the solar motion, assuming a solar velocity of 300 km./sec. I thus obtained the following components of solar motion :—

In the case of Strömberg's theory . .	$X = +$ 96·2	km./sec.
	$Y = +279{\cdot}3$	,,
	$Z = +$ 52·1	,,
In the case of Oort's theory . . .	$X = +172{\cdot}1$	,,
	$Y = +245{\cdot}7$	,,
	$Z =$ 0·0	,,

* *M.N.*, **86**, 686, 1926 ; *Memoirs of Univ. of California*, **7**, 25–29, 1926 ; *Astr. Journal*, **30**, 191–200, 1917 ; *M.N.*, **75**, 526, 1915 ; *Astron. Nachrichten*, **179**, 292–299, 1908 ; *M.N.*, **87**, 122, 1926 ; **88**, 144, 1927.

† *Ap. J.*, **61**, 352–362, 1925.

II. *The Relation between Stellar Motion and Distance from the Galactic Plane*

4. Some time ago Mineur * published a memoir in which he showed that the velocity of the Sun varies when calculated from stars at different distances, both north and south, from the galactic plane. Having the opportunity and feeling the necessity of studying this matter, I repeated Mineur's researches, but with a smaller number of stars. Nevertheless, the following results are quite convincing. The computations were made as follows. The space above and below the galactic plane was divided into layers at intervals of 10 parsecs. The most distant ends only were chosen in larger limits, for the purpose of getting a sufficient number of stars there. The distance of each star from the galactic plane was calculated.

The computations of galactic longitude and latitude of solar motion are as in Table II. The computations are made for giants only, and then for all stars together. It would be of no importance to do the same with dwarfs, because there are relatively very few of them and the distance of single star-groups from the galactic plane is proportionately very small.

The main conclusion to be drawn from these computations is that the direction of solar motion does not change, even when different star-groups at different distances from the galactic plane are employed. Also the velocity of the Sun remains constant in both cases. It is consequently possible to unite all stars which are not in the galactic plane into groups and to study their average motion according to rotation, and it is not necessary to employ stars in or near to the galactic plane only. Mineur also obtained similar results for stars which were at practically the same distances from the galactic plane as those employed in my calculations. For more distant stars he found some change of the direction of stellar motions relative to the Sun, and *vice versa*. It is not impossible that in the highest or lowest layers there may be some stars showing somewhat different motions, because these layers approach the limits of our stellar system, and beyond all doubt these limb-stars succumb not only to the attraction of the centre, but also to the attraction of stars forming the central layer of the galactic system (stars in the galactic plane or in its neighbourhood), which causes considerable inclinations of their orbits relative to the galactic plane. Also, the material used by Mineur for more distant layers is not homogeneous with the material of the layers close to the galactic plane—hence the different values. Therefore his results cannot be accepted as demonstrated.

III. *Distribution of Velocities of the Stars*

5. A number of interesting results can be deduced from the calculated velocities of the stars. The velocities can be analysed in various ways. First, consider the frequency of the velocities. The velocities are divided into groups of intervals 201–210, 211–220, 221–230, 231–240 km./sec., etc., and the results are given graphically in fig. 1.

* *M.N.*, **90**, 789–797, 1930, and *Bull. Astron.*, **6**, 355–397, 1931.

Table II

Limits	Number of Stars	Galactic Co-ordinates of Solar Motion		Velocity of the Sun
		L	*B*	
		(a) *Giants*		
Parsecs		°	°	km./sec.
+81 to > +150	102	39·2	+25·5	19·5
+61 ,, + 80	85	24·4	+18·7	21·2
+51 ,, + 60	76	35·2	+16·6	20·7
+41 ,, + 50	76	13·7	+13·3	22·1
+31 ,, + 40	82	32·7	+19·7	18·7
+21 ,, + 30	77	16·0	+27·3	13·9
+11 ,, + 20	98	44·3	+20·8	17·8
+ 1 ,, + 10	95	29·1	+20·9	17·6
− 1 ,, − 10	107	26·1	+24·9	18·8
−11 ,, − 20	80	23·7	+15·2	18·3
−21 ,, − 30	58	24·7	+20·5	22·0
−31 ,, − 50	87	35·9	+17·8	18·3
−51 ,, − 80	83	23·8	+26·6	12·7
−81 ,, > −100	54	30·1	+32·1	14·3
		(b) *All Stars*		
> +150 to + 81	102	39·2	+25·5	19·5
+ 80 ,, + 61	88	40·6	+20·7	24·6
+ 60 ,, + 51	88	29·5	+ 6·5	32·7
+ 50 ,, + 41	87	25·1	+ 8·9	27·7
+ 40 ,, + 31	105	44·8	+14·8	21·1
+ 30 ,, + 21	129	30·5	+14·2	25·6
+ 20 ,, + 11	181	39·8	+16·7	21·3
+ 10 ,, + 1	233	40·4	+20·2	25·5
− 1 ,, − 10	203	33·0	+10·9	28·4
− 11 ,, − 20	143	27·4	+15·3	25·0
− 21 ,, − 30	79	27·1	+14·7	26·4
− 31 ,, − 40	101	27·9	+16·0	35·8
− 41 ,, − 70	107	39·6	+30·0	15·6
− 71 ,, > −100	75	22·1	+11·3	22·0

This figure shows distinct forms of simple curves, the most symmetrical being the curve for type A. Curves of other types are asymmetric. With increasing mass of the stars of single spectral types –from type A to giants F, G, K, M—the curves become progressively flatter and broader. The dispersion of velocities continually increases. Type B only, formed of stars with high masses, does not agree with this rule, and the velocity-frequencies of this type approach those of type A.

This is valid for both directions of the solar motion. For dwarfs I

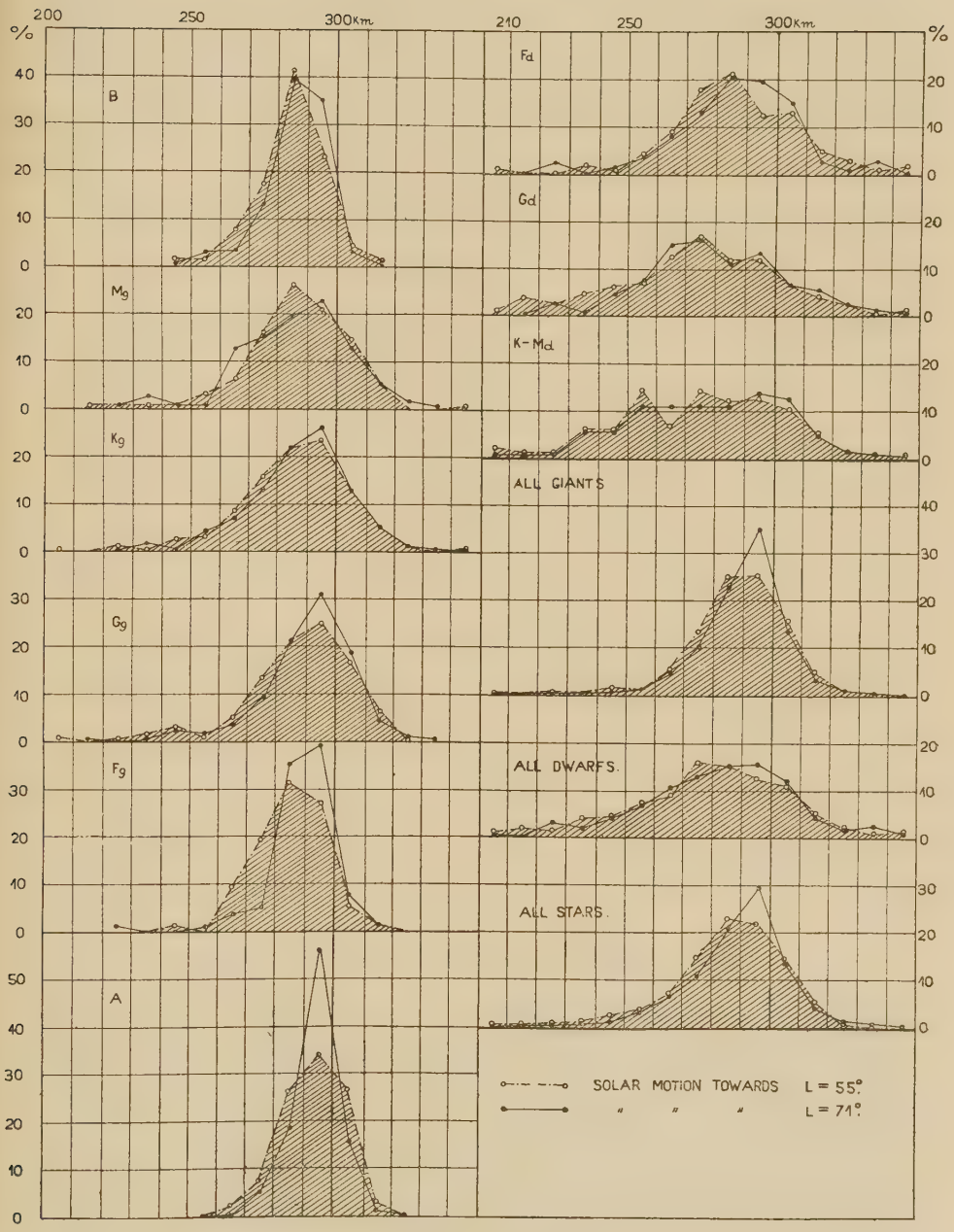


FIG. 1.—*Distribution of Velocities of Stars.*

obtained curves different from those for giants. As the spectral type of dwarfs advances the curve becomes broader, until we observe even two curve-maxima at about 270 and 300 km./sec. ; but the number of dwarfs is insufficient to determine the reality of these results. It is well known that there are more dwarfs in space than giants. Within a space distant 10 parsecs from the Sun there are roughly 15 dwarfs to 1 giant. The number of dwarfs used is thus only 3 per cent. of the number which could

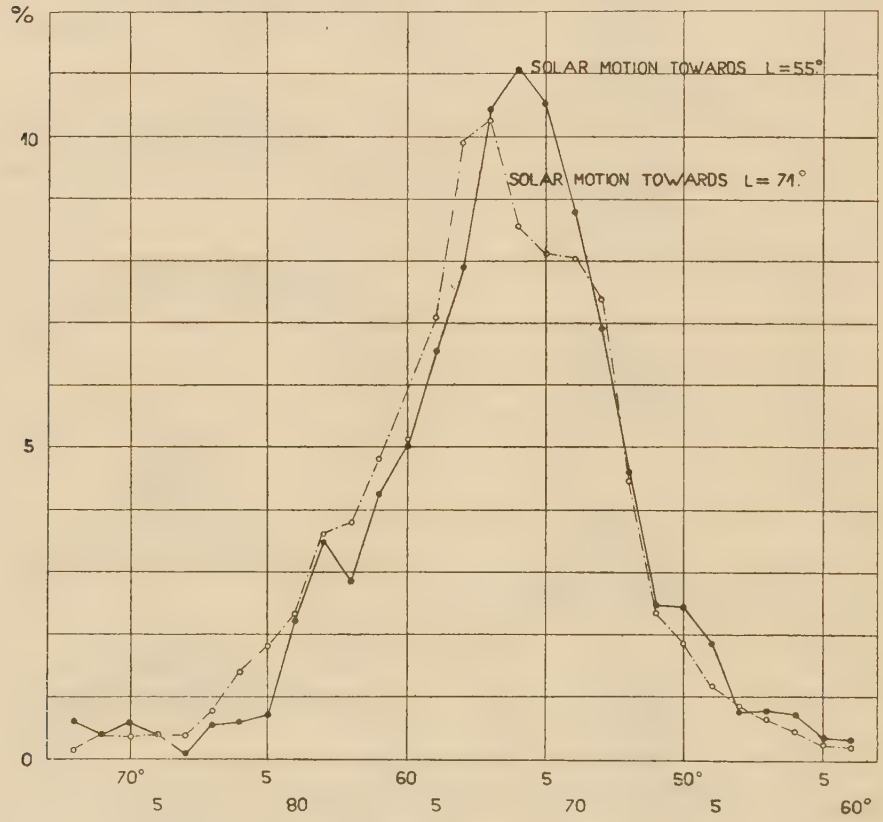


Fig. 2.—*Distribution of the Directions of Motion for Giants.*

be contained in a space appropriate to the 1160 giants considered here. But this number is not so small in reality because the number of giants refers to a much larger volume of space. Nevertheless, the available number of dwarfs can scarcely be higher than 10 per cent. of the total existing within the smaller distance.

In fig. 1 are also given the curves for giants, dwarfs, and all stars, for all spectral types combined. As will be seen from the diagram, velocities of 300 km./sec. are about 5 per cent. less frequent than the most common velocity of giants. The velocity most common to giants is always less than that of the Sun.

6. The study of stellar velocities according to direction is also important. Although stellar motions are not exactly parallel to the galactic plane, the

fixation of a galactic longitude is of the greatest value. I therefore computed the galactic longitude for each direction of stellar motion, assuming in the first case the Sun's motion to be $l=71°$, and in the second case $l=55°$. The results are given graphically in figs. 2 and 3. In both cases the curves for both values of the solar motion are drawn. The curves for giants are very regular, with obvious maxima ; those for dwarfs are very dispersed.

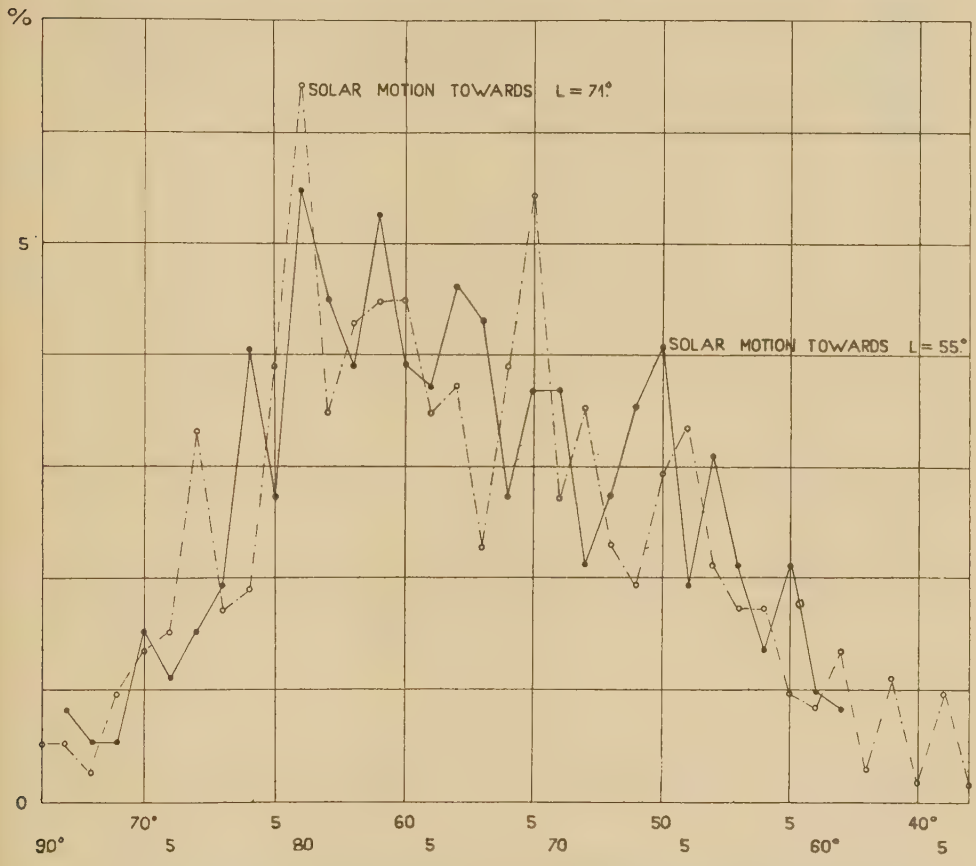


FIG. 3.—*Distribution of the Directions of Motion for Dwarfs.*

The irregularities of the curves for dwarfs are due to insufficient data, but it is nevertheless possible to distinguish the distribution of the motions towards definite directions. In both curves the position of curve-maximum is important. Considering the curves of giants, we see a maximum at 73° in the first and at 56° in the second case. The slope of the curve on both sides of these maxima is significant, being slower towards higher values of the angle. Dwarfs show their maximum at higher values of the angle. The maximum is at 81° in the first and 62° in the second case. The difference between the mean direction of the motion of dwarfs and the solar motion is approximately :

$+10°$ in the first and $+7°$ in the second case.

It would appear that dwarfs are excluded from the rotational circle. The direction of rotation of a large number of dwarfs does not correspond with the direction of solar motion. This is an additional reason for inferring that the curves described by stars round the centre are not circles but ellipses.

7. It is of importance to ascertain which directions are preferred by different stellar velocities. Let us take star-groups of equal-rated space-velocities and find their average angle-values. I took the following group-distributions : < 270 km./sec., 271–280, 281–290, etc., and > 320 km./sec. Computations have been made for both values of solar motion, and the results are given in Tables III and IV.

TABLE III

Solar Motion towards $l = 71°$

A : Giants

< 270 km./sec.	271–280	281–290	291–300	301–310	311–320	Over 320 km./sec.
$74°{\cdot}22_{(129)}$ *	$73{\cdot}21_{(121)}$	$73{\cdot}22_{(278)}$	$72{\cdot}73_{(405)}$	$72{\cdot}37_{(161)}$	$73{\cdot}44_{(41)}$	$77{\cdot}14_{(21)}$

B : Dwarfs

$75°{\cdot}52_{(162)}$	$76{\cdot}96_{(62)}$	$75{\cdot}73_{(77)}$	$75{\cdot}29_{(79)}$	$75{\cdot}37_{(60)}$	$75{\cdot}71_{(24)}$	$76{\cdot}17_{(42)}$

C : All Stars

$75°{\cdot}20_{(291)}$	$75{\cdot}30_{(183)}$	$73{\cdot}76_{(355)}$	$73{\cdot}15_{(484)}$	$73{\cdot}19_{(221)}$	$74{\cdot}28_{(65)}$	$76{\cdot}38_{(63)}$

The tables show a surprising fact. The smallest and highest velocities move away from the direction for galactic rotation. This is true for both giants and dwarfs. Giants show a minimum between 290–310 km./sec., whilst the mean direction-value towards highest and lowest velocities increases rapidly. Dwarfs of the highest velocities show a higher value of the mean direction than those of smaller velocities. The dependence is here almost linear ; with an increase of 10 km./sec. it ascends by $0°{\cdot}25$, and begins for lowest velocities approximately at $74°{\cdot}5$.

This behaviour of star-velocities is somewhat remarkable. If stars, moving towards higher angle-values, have exceptionally low velocities, then it would be comprehensible, because they would move away from the rotational centre and therefore their velocity should be smaller. In spite of this, the velocity is actually low in one and high in another star-group.

* The number enclosed in brackets indicates the number of stars.

Now let us examine the result of a similar analysis, supposing the solar motion towards the galactic longitude $l = 55°$. The result is as in Table IV.

TABLE IV

Solar Motion towards $l = 55°$

A : Giants

Type	<270 km./sec.	271–280	281–290	291–300	301–310	311–320	Over 320
	°	°	°	°	°	°	
B	55·1(14) *	57·8(22)	56·8(53)	56·3(30)	54·7(6)	54·0(2)	
A	57·1(7)	58·4(16)	58·9(56)	55·7(73)	53·4(59)	52·0(6)	53·0(1)
F	56·0(7)	59·2(10)	59·7(16)	55·1(14)	54·3(3)	59·0(1)	
G	59·5(53)	58·7(48)	58·0(77)	55·9(90)	55·0(63)	57·4(26)	58·3(3)
K	59·1(53)	56·7(48)	56·2(68)	56·0(73)	53·9(41)	56·8(17)	57·2(5)
M	58·0(13)	54·6(16)	55·7(26)	56·0(20)	56·1(14)	54·2(6)	55·0(1)
	58·51(150)	57·56(160)	57·45(296)	55·90(300)	54·30(186)	56·24(58)	56·90(10)

B : Dwarfs

F	58·8(40)	57·8(37)	57·8(42)	57·2(25)	52·2(28)	58·2(11)	44·6(12)
G	61·5(66)	57·9(28)	56·8(18)	62·5(18)	54·3(12)	57·0(8)	43·6(8)
K	60·8(57)	56·8(20)	59·7(10)	61·3(16)	56·9(11)	50·8(5)	50·3(4)
M	54·6(8)	56·0(3)	55·1(9)	55·0(4)	55·0(5)	72·3(3)	
	60·32(171)	57·55(88)	57·51(79)	59·60(63)	53·84(56)	58·04(27)	45·21(27)

C : All Stars

	59·48(321)	57·56(248)	57·46(375)	56·54(363)	54·19(272)	56·81(85)	48·65(34)

Considering the table for giants, we can see stars with lowest and highest velocities moving towards higher values of galactic longitude, but the increase of these values is small. For dwarfs, those with smallest velocities move towards high, and stars with highest velocities move towards lowest values of the galactic longitude. For all stars together the relation is the same as for dwarfs. This means that stars moving towards the smallest galactic longitudes show higher velocities, whilst stars moving away from the galactic centre show smaller velocities. Why this dependency is more noticeable with dwarfs than with giants I will explain later. Be it said once more: this relation is apparent only when we presume the Sun to move towards the galactic longitude $l = 55°$, and now it can be understood why it was so important not to decide upon one or other of these directions of solar motion.

* The number enclosed in brackets indicates the number of stars.

IV. *Masses of the Stars*

8. The masses of stars, here used, have been calculated by Eddington's method, which determines the mass of a star when its absolute bolometric magnitude is known. As only photographic magnitudes were known, it was necessary to change them first into visual and then into bolometric magnitudes. Colour indices used are those given by Davidson and Martin.* Eddington's table † giving the relation between the mass and the absolute bolometric magnitude is calculated for the effective temperature $T_e = 5{,}200°$ C. This is the temperature of spectral type G. For the other spectral types the term to be added to the absolute magnitude is

$$-2 \log_{10}\left(\frac{T}{5{,}200}\right).$$

The following values of the temperatures of individual spectral types were used :—

Type	Temperature	Temperature Term
B	19,000° C.	−1·126
A	10,000	−0·568
F	7,000	−0·258
G	5,200	0·000
K	3,900	+0·230
M	3,000	+0·478

The distribution of the masses in different spectral types is given in Table V. Of course such a table is usually applied to the distribution of absolute magnitudes, in which the main sequence is a straight line ; while the main sequence in the mass-diagram is shown to be a hyperbolic curve falling suddenly from type B to type A.

V. *The Relation between Masses and Rotational Velocities of Stars*

9. It was early known that stars of different spectral types show different velocities. It is important to find whether there exists some relation between the rotational velocities of stars and their masses. By dividing stars of the same spectral type into groups having the same mass, and investigating their velocities, we can see whether such a relation exists or not. Such groups were chosen in the following way. Stars with the mass of 5·0 ⊙ and higher gave one group ; the succeeding groups were composed of stars 4·9–4·5 ⊙, and so on. It was not possible to conform to this rule in every case—smaller limits had to be taken for dwarfs. The results of this research are given in Table VI.

As the relation between the masses of stars and their velocities is not shown clearly enough in the table, this dependence for both values of the solar

* *M.N.*, 84, 430, 1924.

† *Internal Constitution of the Stars*, chap. vii (Cambridge University Press), 1926 ; *M.N.*, 84, 308, 1924.

TABLE V

Distribution of Masses

Masses (⊙ = 1)	B04	B59	A04	A59	F04	F59	G04	G59	K04	K59	Ma	Mb	Mc
> 8·5	8	...	...	...	...	6	6	...	...	2	2	3	2
8·5 – 8·4	3												
8·3 – 8·2	...	...	...	...	...	1							
8·1 – 8·0	7												
7·9 – 7·8													
7·7 – 7·6	1												
7·5 – 7·4	...	...	...	...	...	...	1						
7·3 – 7·2													
7·1 – 7·0	3	...	...	...	...	...	1						
6·9 – 6·8	...	...	...	...	...	...	...	...	...	...	1	1	
6·7 – 6·6	...	...	...	...	...	...	...	...	...	...	...	1	
6·5 – 6·4	2	...	...	...	...	...	1	...	...	...	3		
6·3 – 6·2	...	...	...	...	...	1	2	...	...	...	...	...	1
6·1 – 6·0	1	...	...	...	...	1	...	1	...	...	2	1	
5·9 – 5·8	5	1	...	...	...	1							
5·7 – 5·6	...	...	...	...	...	...	...	1	...	...	...	1	
5·5 – 5·4	...	...	...	...	...	1	1	1	...	...	4	2	
5·3 – 5·2	4	...	...	...	...	...	1	1	...	...	3	2	1
5·1 – 5·0	...	3	...	...	...	...	1	...	...	...	2	3	
4·9 – 4·8	10	5	...	...	...	2	1	...	...	...	4	3	1
4·7 – 4·6	...	...	...	...	...	1	1	1	...	...	4	1	1
4·5 – 4·4	4	3	...	...	...	1	...	1	...	...	5	2	
4·3 – 4·2	2	1	...	...	...	...	1	...	1	1	6	4	
4·1 – 4·0	5	7	...	...	...	1	3	1	1	...	12	2	
3·9 – 3·8	...	13	...	...	2	...	2	2	5	...	4	3	
3·7 – 3·6	...	7	...	...	...	...	2	4	1	1	1	1	
3·5 – 3·4	...	3	...	...	...	...	1	7	19	11	3	1	
3·3 – 3·2	...	8	2	...	1	2	3	13	41	26	2		
3·1 – 3·0	...	9	...	1	1	...	1	12	49	15	...	1	
2·9 – 2·8	...	11	5	...	1	...	7	19	36	4	1		
2·7 – 2·6	...	2	10	...	1	2	2	27	46				
2·5 – 2·4	...	...	44	2	1	2	8	44	11				
2·3 – 2·2	...	...	34	...	3	3	6	46	16				
2·1 – 2·0	...	...	23	...	6	1	2	30	7				
1·9 – 1·8	...	...	32	9	7	4	5	41	5				
1·7 – 1·6	...	...	16	32	11	2	2	21	8				
1·5 – 1·4	...	...	...	7	26	6	9	14	4				
1·3 – 1·2	...	...	1	...	10	25	8	5	2				
1·1 – 1·0	...	...	...	...	24	81	30	10	8				
0·9 – 0·8	...	...	1	...	...	12	36	19	9				
0·7 – 0·6	...	...	...	...	...	3	17	36	66	15	6		
0·5 – 0·4	...	...	...	...	...	...	...	...	1	20	22		
< 0·3	...	...	...	...	...	...	...	...	...	...	2		

TABLE VI

Type	Number of Stars	$\bar{M}$ ($\odot=1$)	Velocities of Groups		Mean Direction of Motion		Limits of Masses
			Solar Motion towards $l=71°$	Solar Motion towards $l=55°$	Solar Motion towards $l=71°$	Solar Motion towards $l=55°$	
			km./sec.	km./sec.	km./sec.	km./sec.	
B	40	7·55	284·8 ± 13·6	283·8 ± 14·6	70·9 ± 4·1	56·5 ± 3·2	>5·0
	22	4·74	287·3 ± 10·1	284·9 ± 7·5	73·6 ± 1·7	56·8 ± 1·7	4·9 – 4·5
	66	3·56	287·6 ± 10·0	286·2 ± 10·0	73·3 ± 2·9	56·5 ± 2·9	<4·4
Mg	34	6·97	282·9 ± 20·8	283·2 ± 22·0	72·3 ± 8·9	55·6 ± 8·4	>5·0
	62	4·17	289·6 ± 19·3	288·8 ± 16·6	72·7 ± 6·1	55·9 ± 4·6	<4·9
Kg	71	3·58	290·5 ± 17·7	288·8 ± 20·2	73·4 ± 6·7	56·9 ± 6·6	>3·3
	66	3·16	286·2 ± 23·6	284·5 ± 24·9	73·3 ± 6·7	56·4 ± 6·0	3·2 – 3·1
	76	2·93	282·6 ± 24·8	281·5 ± 24·9	73·3 ± 6·0	56·1 ± 5·6	3·0 – 2·8
	93	2·39	287·5 ± 20·6	285·6 ± 21·5	73·4 ± 7·0	56·7 ± 7·0	<2·7
Gg	75	4·83	287·6 ± 27·5	286·8 ± 21·6	73·3 ± 7·3	56·5 ± 8·5	>3·0
	70	2·67	289·4 ± 20·1	288·4 ± 24·0	73·1 ± 5·9	56·8 ± 5·4	2·9 – 2·4
	121	2·24	289·1 ± 16·5	286·2 ± 19·7	74·3 ± 6·9	57·5 ± 6·1	2·4 – 2·0
	94	1·74	291·0 ± 22·3	287·3 ± 22·6	74·8 ± 6·6	58·2 ± 6·4	<1·9
Fg	38	3·71	284·5 ± 18·9	282·3 ± 15·9	74·0 ± 5·8	56·8 ± 6·1	>2·0
	13	1·82	294·2 ± 9·0	289·7 ± 10·6	75·2 ± 4·8	58·8 ± 3·8	<1·9
A	25	2·70	293·3 ± 8·3	290·3 ± 10·3	73·6 ± 3·8	57·0 ± 3·7	3·0 – 2·5
	96	2·24	293·9 ± 9·9	295·1 ± 11·6	71·4 ± 3·2	55·2 ± 2·9	2·4 – 2·0
	98	1·61	293·9 ± 8·6	293·1 ± 10·0	72·7 ± 4·3	56·5 ± 4·3	<1·9
Fd	58	1·45	291·0 ± 16·0	291·0 ± 16·9	73·1 ± 5·7	56·6 ± 5·5	>1·3
	128	1·07	284·0 ± 34·5	281·8 ± 33·5	75·1 ± 14·6	57·0 ± 14·5	1·2 – 1·0
	14	0·84	289·5 ± 36·7	282·7 ± 33·9	77·1 ± 12·8	60·2 ± 13·2	<0·9
Gd	82	1·00	280·9 ± 27·3	275·0 ± 27·5	76·3 ± 11·1	59·5 ± 10·2	1·0 – 0·9
	77	0·72	273·1 ± 33·7	271·3 ± 41·0	76·7 ± 16·0	59·4 ± 15·9	0·8 – 0·7
Kd	22	1·04	274·8 ± 30·0	271·6 ± 29·8	74·9 ± 12·2	57·4 ± 12·4	>0·8
	64	0·70	274·3 ± 31·3	268·6 ± 30·5	76·6 ± 12·3	59·2 ± 12·2	0·7
	39	0·51	278·5 ± 27·0	273·2 ± 26·6	76·4 ± 9·6	59·2 ± 9·7	0·6 – 0·4
Md	18	0·54	279·3 ± 30·8	279·2 ± 25·1	73·4 ± 11·0	56·2 ± 11·8	0·7 – 0·5
	14	0·39	287·9 ± 37·2	284·0 ± 26·1	73·9 ± 12·4	57·2 ± 13·0	0·4 – 0·2

motion is shown graphically in fig. 4. As shown in this figure, in the giant branch the velocity of giants increases with decreasing mass ; stars of the type K are an exception.

It is important to call attention to the dispersion of velocity of single groups, expressed by the values ± placed next to the mean velocities of the

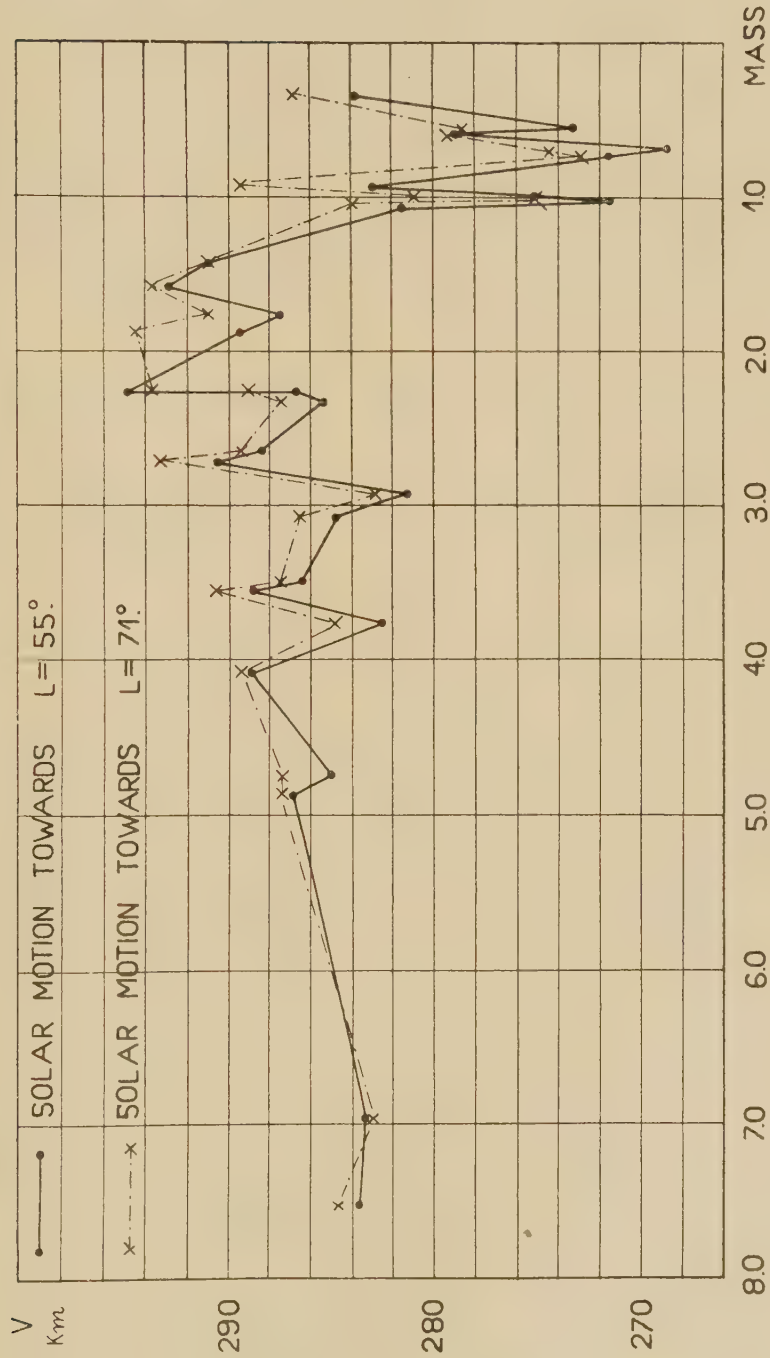


FIG. 4.—*Relation between the Masses and Velocities of Stars.*

groups and defined according to formulæ for the mean error. The smallest differences of velocity—or smallest velocity-dispersion—are to be seen at type A, which shows also the highest stellar velocity. Going down from type A over the giants F, G, K, M we see the dispersion-velocity increasing and falling again at slow-moving stars of type B.

For dwarfs the contrary is the case; with decreasing mass the velocity decreases. Only dwarfs of type M do not satisfy this rule and show higher velocities, as shown in fig. 4.

Stars of types A and F, and to some extent also G, have the highest mean-velocities. Fig. 4 shows very clearly that those are stars, having approximately 2·0 of the solar mass. Assuming the relation between mean-velocity and mean-mass to be linear, determined by the equation

$$\bar{v} = a + b\bar{M},$$

we obtain for giants (supposing the weight of each relation between v and $\bar{M}$ equal to 1):

Solar motion towards $l = 71°$	Solar motion towards $l = 55°$
$\bar{v} = 293{\cdot}61 - 1{\cdot}42\bar{M}$	$\bar{v} = 290{\cdot}89 - 1{\cdot}11\bar{M}$
$\pm 1{\cdot}55 \pm 0{\cdot}42$	$\pm 1{\cdot}58 \pm 0{\cdot}43$

and for dwarfs

$\bar{v} = 263{\cdot}23 + 18{\cdot}33\bar{M}$	$\bar{v} = 253{\cdot}62 + 24{\cdot}45\bar{M}$
$\pm \quad 6{\cdot}14 \pm 6{\cdot}10$	$\pm 8{\cdot}71 \pm 8{\cdot}66.$

These straight lines cut each other at a point of mean mass 1·74 ⊙ and velocity $v = 291{\cdot}14$ km./sec. for the first value of solar motion $l = 71°$; and at $M = 1{\cdot}46$ ⊙ and $v = 289{\cdot}30$ km./sec. for the second value.

Applying to each star-group a weight, corresponding to the number of stars forming this group, we obtain for giants the following equations:

Solar motion towards $l = 71°$	Solar motion towards $l = 55°$
$\bar{v} = 292{\cdot}72 - 1{\cdot}15\bar{M}$	$\bar{v} = 290{\cdot}85 - 1{\cdot}09\bar{M}$
$\pm 23{\cdot}11 \pm 0{\cdot}85$	$\pm 30{\cdot}33 \pm 1{\cdot}12$

and for dwarfs

$v = 265{\cdot}93 + 15{\cdot}79\bar{M}$	$v = 259{\cdot}52 + 19{\cdot}25\bar{M}$
$\pm 34{\cdot}95 \pm \quad 3{\cdot}84$	$\pm \quad 41{\cdot}94 \pm \quad 4{\cdot}61.$

The point of intersection of both straight lines is defined by the average mass $M = 1{\cdot}58$ ⊙ and velocity $v = 290{\cdot}89$ km./sec. for the solar motion towards $l = 71°$; and by the co-ordinates $M = 1{\cdot}54$ ⊙ and $v = 289{\cdot}17$ km./sec. for the second value. All these determinations of the mean mass with the highest mean velocity are within the limits of 1·5 – 2·0 of the Sun's mass and about 290 km./sec.

I have examined until now the mean velocities of star-groups of different spectral types. Now let us consider the mean velocities of stars, formed from allspectral types, and divided into groups $M \geq 5{\cdot}0$⊙; 4·9 – 4·5⊙; 4·4 – 4·0⊙; 3·9 – 3·5⊙; 3·4 – 3·0⊙, etc., until we reach the group 1·4 – 1·0 ⊙. Thence

divide the dwarfs into groups having masses 0·9 – 0·8⊙, 0·7 – 0·6⊙, and 0·5 – 0·2 ⊙. Values of the mean velocities and of the mean directions of motion calculated for these groups are given in Table VII.

Table VII

Stellar Masses	Number of Stars	$\bar{M}$ (⊙ = 1)	Average Velocity		Average Direction of Motion	
			$l = 71°$	$l = 55°$	$l = 71°$	$l = 55°$
			km.	km.	°	°
≧ 5·0⊙	101	7·60	284·1	283·5	72·3	56·0
4·9 – 4·5	49	4·72	287·5	285·3	74·0	57·2
4·4 – 4·0	50	4·14	288·6	288·3	72·7	55·9
3·9 – 3·5	71	3·71	290·5	287·9	73·9	57·3
3·4 – 3·0	208	3·19	286·3	285·6	73·0	56·2
2·9 – 2·5	204	2·73	288·9	286·4	73·8	57·2
2·4 – 2·0	257	2·24	290·0	289·0	72·9	56·3
1·9 – 1·5	233	1·71	292·5	290·5	73·5	57·0
1·4 – 1·0	225	1·13	283·3	280·8	74·4	56·9
0·9 – 0·8	77	0·85	280·9	274·4	76·6	59·6
0·7 – 0·6	143	0·68	272·8	269·3	76·6	59·2
0·5 – 0·2	48	0·44	283·6	279·9	75·1	58·3

In Table VII the increase of the velocity of giants according to the decrease of mean mass of the groups is readily seen. The irregularity between masses 4·5 – 3·0 ⊙ concerns stars of spectral type K, and is inexplicable at present. For dwarfs it is possible to observe a regular decrease of velocity with decrease of the mass, except for the group of lightest stars, which show higher average velocity. We shall examine which of the two branches of the relations mentioned corresponds to reality. But first I will consider the relation between mass and the direction of stellar motion. Star-groups, as in Table VII, show that dwarfs mostly move away from the rotational centre ; the mean direction-values of these stars are greater than 75° and 58° respectively. This rapid increase of the direction of motion occurs after the star-group of type A—that is, at the mass 1·0 ⊙.

Both of these relations, unknown till now—the relation between the mean mass of stars, the average velocity and the mean direction of stellar motion—are of such importance that they should be further studied. Discussion of these results is important, because dwarfs so far available for computation have been mainly those with large proper motions.

VI. *Discussion of the Proper Motions of Stars*

10. Before continuing this discussion, I will consider whether there is a better relation between stellar mass and space-velocity. I will try to

determine the relation between proper motion and space-velocity—if any exists. For this purpose stars were divided into groups of proper motions μ_α, μ_δ, ascending within limits $\pm 0''.001 - \pm 0''.010$; $\pm 0''.011 - \pm 0''.020$; $\pm 0''.021 - \pm 0''.030$; $\pm 0''.031 - \pm 0''.040$ . . . etc., and for these groups mean space-velocities and mean parallaxes were investigated. The chief result of these studies is the possibility of replying to a series of questions unanswered up to the present.

Single spectral types do not clearly demonstrate the relation, so I calculated mean velocities for giants and dwarfs of all spectral types. For these groups mean parallaxes and frequencies have also been found. Results of these computations are given for giants in Table VIII, for dwarfs in Table IX.

Table VIII

Giants

Proper Motion	Number of Stars	Mean Group-velocities (km./sec.)		Mean Parallax	Frequency of Stars in per cent.	Mean Direction of Motion
		$\odot l = 71°$	$\odot l = 55°$			Solar Motion towards $l = 55°$
$\pm 0''.001 - \pm 0''.010$	168	293·32	292·42	$0''.009_7$	14·53	55·99
$\pm 0''.011 - \pm 0''.020$	205	291·56	291·51	$0''.010_0$	17·73	55·46
$\pm 0''.021 - \pm 0''.030$	157	291·59	290·96	$0''.012_6$	13·58	56·06
$\pm 0''.031 - \pm 0''.040$	95	292·45	292·11	$0''.015_9$	8·22	56·07
$\pm 0''.041 - \pm 0''.050$	79	290·39	289·51	$0''.016_3$	6·83	56·25
$\pm 0''.051 - \pm 0''.060$	70	289·34	289·17	$0''.016_5$	6·06	56·00
$\pm 0''.061 - \pm 0''.070$	53	286·40	285·83	$0''.015_6$	4·58	55·79
$\pm 0''.071 - \pm 0''.080$	48	286·63	284·06	$0''.020_5$	4·15	57·27
$\pm 0''.081 - \pm 0''.090$	41	285·98	283·20	$0''.018_7$	3·55	57·34
$\pm 0''.091 - \pm 0''.100$	29	281·14	273·10	$0''.017_3$	2·51	60·76
$\pm 0''.101 - \pm 0''.200$	137	285·12	282·13	$0''.025_8$	11·85	57·58
$\pm 0''.201 - \pm 0''.300$	34	260·41	259·24	$0''.025_4$	2·94	56·27
$\pm 0''.301 - \pm 0''.400$	19	283·11	269·26	$0''.036_9$	1·64	64·58
$\pm 0''.401 - \pm 0''.500$	7	291·43	289·57	$0''.069_9$	0·61	57·57
$\pm 0''.501 - \pm 0''.600$	4	302·75	280·50	$0''.032_5$	0·35	74·75
$\pm 0''.601 - \pm 0''.700$	4	249·25	248·75	$0''.042_0$	0·35	54·75
$\pm 0''.701 - \pm 0''.800$	2	301·50	291·00	$0''.044_0$	0·17	62·00
$\pm 0''.801 - \pm 0''.900$	...	...	...	...	...	...
$\pm 0''.901 - \pm 1''.000$	...	...	...	...	...	...
$> \pm 1''.000$	4	274·75	271·00	$0''.135_8$	0·35	61·33

The frequency of stars in the single groups in Tables VIII and IX shows a maximum for giants of 17·73 per cent. within the limits $\pm 0''.011 - \pm 0''.020$. The frequency curve for giants is quite regular. For dwarfs

TABLE IX

Dwarfs

Proper Motion	Number of Stars	Mean Group-velocities (km./sec.)		Mean Parallax	Mean Direction of Motion
		$\odot l=71°$	$\odot l=55°$		Solar Motion towards $l=55°$
″ ″				″	
± 0·001 – ± 0·010	2	298·0	289·5	0·014	...
± 0·011 – ± 0·020	3	291·0	297·7	0·024	...
± 0·021 – ± 0·030	5	298·8	292·8	0·028	...
± 0·031 – ± 0·040	1	303·0	301·0	0·021	...
± 0·041 – ± 0·050	2	301·5	302·0	0·034	...
± 0·051 – ± 0·060	5	297·8	299·0	0·033	...
± 0·061 – ± 0·070	3	294·3	294·3	0·039	...
± 0·071 – ± 0·080	3	286·3	287·7	0·045	...
± 0·081 – ± 0·090	5	289·4	284·4	0·031	...
± 0·091 – ± 0·100	6	297·0	289·7	0·027	...
± 0·101 – ± 0·200	80	288·9	285·9	$0·040_7$	57·5
± 0·201 – ± 0·300	69	285·1	278·8	$0·041_3$	59·8
± 0·301 – ± 0·400	86	278·6	277·4	$0·055_7$	58·3
± 0·401 – ± 0·500	53	271·1	264·4	$0·052_4$	57·8
± 0·501 – ± 0·600	45	269·9	272·4	$0·052_2$	55·5
± 0·601 – ± 0·700	28	281·8	278·0	$0·067_2$	62·1
± 0·701 – ± 0·800	20	267·0	271·8	$0·065_8$	52·0
± 0·801 – ± 0·900	19	265·4	264·1	$0·063_2$	55·8
± 0·901 – ± 1·000	11	262·8	264·1	$0·079_5$	54·2
> ± 1·000	64	272·7	269·4	$0·142_5$	60·2

there is a scarcity of stars between ± 0″·001 – ± 0″·100, so that the frequency curve within those limits is unknown for dwarfs.

The important result of these computations (according to the investigations of Prof. van Rhijn) * is, that *giants and dwarfs with large proper motions show only very small space-velocities.* This is shown more clearly with dwarfs, because there are a great many of these stars with μ_α, μ_δ over ±0″·100. It is also visible from the values of mean velocities computed for groups with μ_α and μ_δ ± 0″·100. This result is real. By comparing the results for the groups μ_α and μ_δ = ± 0″·000 – ± 0″·100, ± 0″·101 – ± 0″·200, ± 0″·201 – ± 1″·000, we may state a second important fact: *for the same values of the proper motions, the mean velocities of dwarfs are always higher than the mean velocities of giants.* (See Table X.)

But the most important result of one of these computations (supposing the Sun moving towards $l=55°$) is that star-groups, formed by *stars with*

* *Publ. Astron. Laboratory at Groningen*, **37**, 12–13, 1925.

large proper motions, move towards higher values of galactic longitude. This relation is very noticeable with giants (see Table VIII) but not so with dwarfs, owing to the lack of sufficient material (Table IX). Nevertheless, it can be seen from Table X that the mean direction of stellar motion tends towards higher values of galactic longitude as the proper motions of the stars increase. This agrees with the fact determined in Section III, that stars with small space-velocities move towards the apogalacticum. The discussion of proper motions shows that it concerns stars with large proper motions only.

Table X

Proper Motions μ_α, μ_δ	Giants			Dwarfs			Direction of Motion (Gal. Long.)	
	Number of Stars	Solar Motion towards		Number of Stars	Solar Motion towards		Solar Motion towards $l=55°$	
		$l=71°$	$l=55°$		$l=71°$	$l=55°$	Giants	Dwarfs
		km./sec.	km./sec.		km./sec.	km./sec.	°	°
± 0″·001 – ± 0″·100	945	290·6	289·7	35	295·1	292·6	56·2	57·0
± 0″·101 – ± 0″·200	137	285·1	282·1	80	289·9	285·9	57·6	57·5
± 0″·201 – ± > 1″·000	74	272·7	266·8	395	276·0	272·8	59·9	58·2

Since 93 per cent. of the dwarfs considered are stars whose proper motions are larger than ± 0″·100, which show the above-mentioned fact of small space-velocity, it is easy to comprehend that the curve showing the relation between the mean mass and mean velocity of star-groups is broken (just at reaching the limit between giants and dwarfs) and falls according to the mass at 1·5⊙ (fig. 4). Discussion of the proper motions shows very well that velocities of dwarfs are *de facto* higher than those of giants, and were we able to utilise all dwarfs existing within a sphere with a radius of 1000 parsecs for our statistical investigations, the lack of dwarfs with proper motions between ± 0″·000– ± 0″·100 would be obviated. The relation between star velocity and mass would probably be given by a straight line which we may imagine produced from the straight line of giants.

VII. *The Discussion of Space-velocities of the nearest Dwarfs*

11. A discussion of the space-velocities of the nearest dwarfs may confirm this last assertion. There are 82 stars within a sphere of radius 10 parsecs ; 3 are from the type A, 7 from F (dwarfs), 22 from G (only 2 of these are giants), 32 from K (one giant), 18 from M (all dwarfs). There are 6 giants (7·3 per cent.) and 76 dwarfs (92·7 per cent.) within a distance of 10 parsecs. Owing to the impossibility of asserting that these figures

include all the dwarfs within this distance there may possibly be errors, but such errors could only be in the direction of underestimating the percentage of dwarfs. If this could be applied to a greater space, with the exception of stars of the type B (which are contained in relatively small numbers in a given volume of space) and of dwarfs which are in the later spectral types, we should discover a great deal.

The mean velocities, mean directions of motion, and mean masses of dwarfs in a sphere with a radius of 10 parsecs (the proper motions of which are naturally higher than $\pm$ 0″·100) are given in Table XI.

Table XI

Type	Solar Motion towards $l=71°$		Solar Motion towards $l=55°$		Mass (⊙ = 1)	Number of Stars
	Velocity	Direction of Motion	Velocity	Direction of Motion		
	km./sec.	°	km./sec.	°		
F	289·4	72·0	297·1	55·6	1·04	7
G	282·0	73·4	282·0	56·6	0·71	20
K	286·5	75·4	282·3	57·8	0·62	31
M	286·6	72·9	283·9	57·2	0·42	18

The mean velocities of single groups of dwarfs (supposing the Sun's motion towards $l=71°$) are never higher than the mean velocity of type A. In the case of the solar motion $l=55°$, the mean velocity of stars of type F only is higher and equal to $v=297·1$ km./sec. Velocities of the other spectral types are smaller, and it could be concluded from this that the nearest dwarfs move with lower velocities than the mean velocities of type A. But this conclusion is incorrect for the following reasons. The first is the distribution of the velocity-frequencies of the 76 nearest dwarfs. This distribution is given in fig. 5.

The maximum frequency is in both cases between 290–300 km./sec., as it should be. The interpretation of the sudden falling of both curves towards high velocities is that high velocities cannot exist in a rotation of stars round the centre, but only low velocities for some reason. With decreasing mass of dwarfs the mean direction of motion of the stars increases. The discrepancy at type M is not very important, taking into account the fact that the hypothesis of solar motion towards $l=55°$ may be more correct. The value 57°·8 for dwarfs of type K is probably higher than the real value, and the value 57°·2 for dwarfs of type M is not very different from this value of type K. Mean values for both types K–M are :

	Solar Motion towards	
	$l=71°$	$l=55°$
Mean mass : 0·55 ⊙. Mean direction of motion :	$a_{71}=74°·5$	$a_{55}=57°·6$

It was shown that higher values of the angle α belong (presuming the solar motion towards $l = 55°$) to stars with small velocities, and smaller values of the angle α belong to stars with high velocities. This circumstance is easily explicable. Stars moving in a direction which is higher than $l = 56°$ move towards the *apogalacticum*, stars moving towards the *perigalacticum* move towards smaller values of the angle α—they move necessarily more rapidly. Nobody will deny that the orbits of stars around the centre are ellipses, considering the great dispersion of the directions of the velocities.

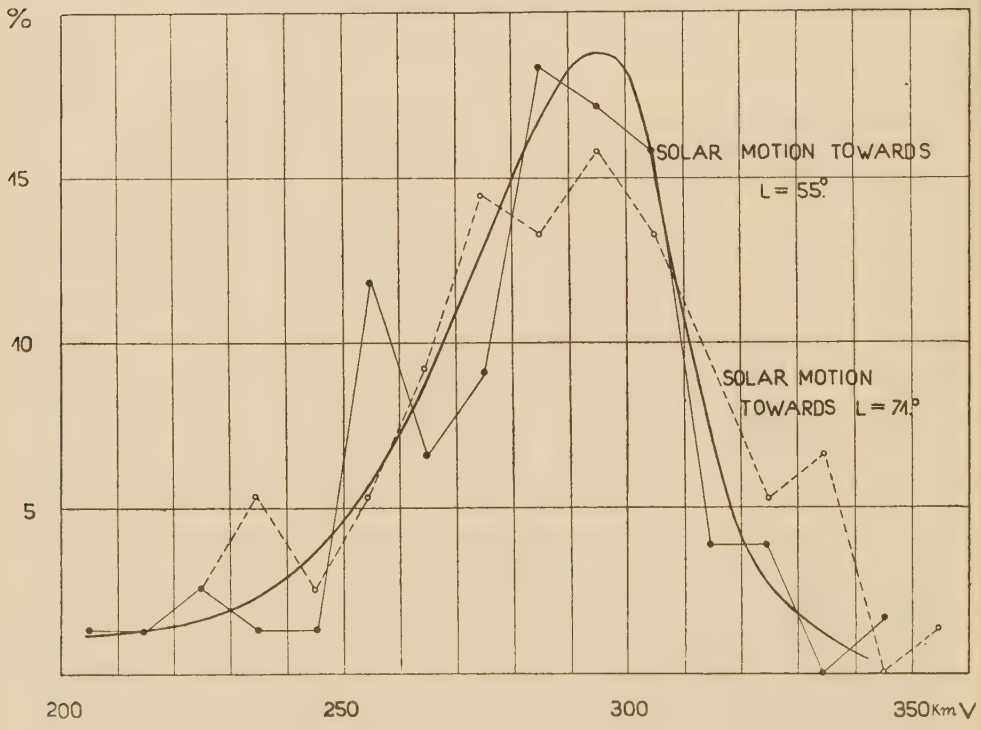


FIG. 5.—*Frequency of Velocities for the nearest Dwarfs.*

Despite the number of dwarfs moving with small velocities there are still many dwarfs with comparatively high velocities. Studying only dwarfs, the velocities of which are within the limits of 270–320 km./sec. (these limits correspond nearly to the maximum of frequencies for dwarfs), we obtain values as in Table XII.

Presuming solar motion to be towards $l = 71°$, we observe a diminution of mean velocity, but presuming that the Sun moves towards $l = 55°$ (and this is far more important), we are able to deduce an almost linear increase of mean velocities of single star-groups. Because the value for solar motion $l = 55°$ is better founded, it is probable that values of the velocity calculated for this case are supported sufficiently.

Thus the dependence of stellar velocity upon mass appears to be established. Instead of the broken line we obtain a linear relation which can be expressed thus : with decreasing mass the rotational velocity of stars increases.

Table XII

Type	Solar Motion towards $l=71°$			Solar Motion towards $l=55°$		
	Number of Stars	Mean Velocity	Mass	Number of Stars	Mean Velocity	Mass
		km./sec.			km./sec.	
F, G	16	294·2	0·97	14	292·6	0·95
K	20	293·1	0·60	21	293·8	0·63
M	10	290·8	0·45	14	294·3	0·42

VIII. *New Observational Evidence of the Rotation*

12. A simple proof of stellar motion round the galactic centre could be given if the following state, which must necessarily exist, could be made apparent. If we assume stellar motion to be circular, it is evident that the direction of a tangent at any point on this circle will have a different angle-value in relation to a fixed point. Taking account only of stars which are in our environment (within a distance of 1000 parsecs), such stars, being immediately within a space which lies in the direction of solar rotational motion ($l=55°$ or $71°$), may be in the orbit preceding the Sun, and must show a direction towards a point whose galactic longitude will be smaller than that of another point towards which stars in the immediate vicinity of the Sun, or in the orbit following the Sun, would move. And I now furnish the evidence of this.

13. First, it will be necessary to observe the present position in space of the stars used. This will be useful also for another purpose. The distribution of the stars with regard to direction is shown in figs. 6A and 6B.

In all these distributions the majority of stars appear between galactic longitudes 341–90–180°, and there are few between 180–270–340°. This corresponds to the fact that such scanty data are available for stars of the southern hemisphere.

We may assume a higher number of stars in the orbit "preceding" the Sun, and a lower number in the orbit "following" the Sun. Now there is the question of choosing and grouping these stars. Taking all stars within limits of galactic longitude $71° \pm 20°$ and $55° \pm 20°$ respectively involves a much smaller area for the nearer stars than for the more distant stars, but this fact is not of great importance because a larger number of distant stars was obtained.

One group was made in each of the directions towards $251° \pm 20°$ and $235° \pm 20°$ respectively, and was formed in the first case by stars distant 50–100 parsecs, together with 5 stars at a distance greater than 100 parsecs; in the case $235° \pm 20°$ the group was formed by another number of stars, there being 23 stars from 50–100 parsecs and 6 stars over 100 parsecs.

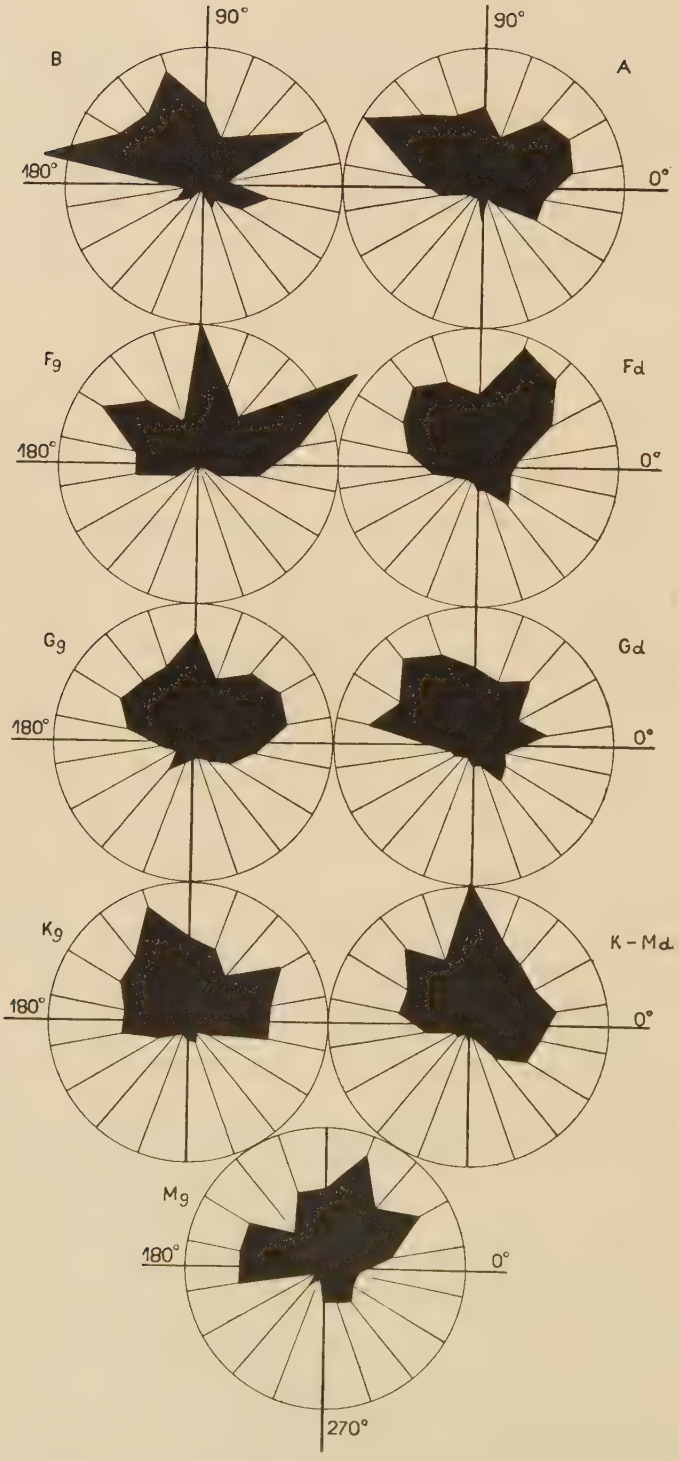


FIG. 6A.—*Percentage Distribution of the Stars (Single Spectral Types) in different Galactic Longitudes.*

The second group was made up of stars distant 0–50 parsecs towards $251^\circ \pm 20^\circ$ and $235^\circ \pm 20^\circ$ respectively, and of stars distant 0–50 parsecs towards $71^\circ \pm 20^\circ$ and $55^\circ \pm 20^\circ$ respectively. This is a group of stars which are very near to the Sun, and its mean direction of motion gives the tangent of rotation in the neighbourhood of the Sun. The third group was made up of stars distant 50–100 parsecs towards $71^\circ \pm 20^\circ$ and $55^\circ \pm 20^\circ$ respectively. The fourth group was formed of stars distant 100–200 parsecs towards $71^\circ \pm 20^\circ$ and $55^\circ \pm 20^\circ$ respectively, and the last group was of stars distant more than 200 parsecs towards these directions. The first

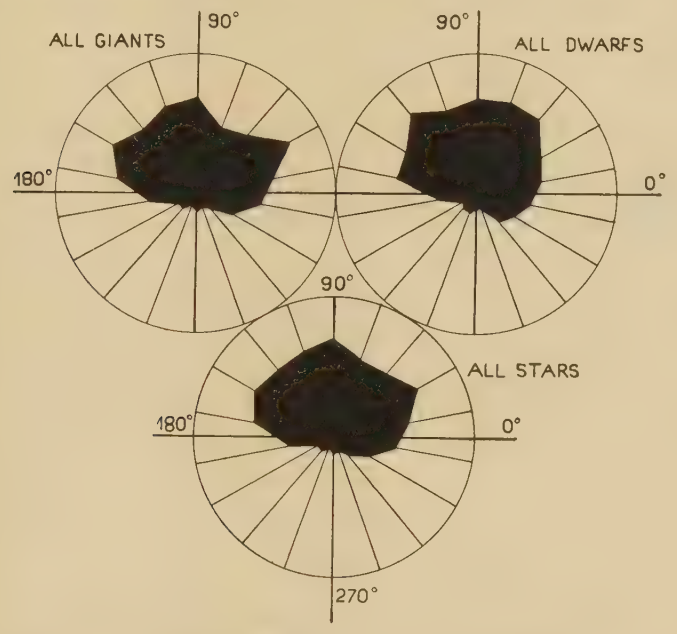


Fig. 6b.—*Percentage Distribution of Giants, Dwarfs and all Stars in different Galactic Longitudes.*

group is formed of stars which are in the orbit "following" the Sun, the second group is in the closest proximity to the Sun, and the other groups are all in the orbit "preceding" the Sun.

Distances of stars, here discussed, are distances parallel to the galactic plane. It has been shown already that the motion of stars at different distances from the galactic plane is similar to the motion of stars in the galactic plane. So there was no necessity to take into account the real distances of stars, but their projections, $R \cos b$, on the galactic plane had to be taken into consideration, where R is the real distance of a star in parsecs and b its galactic altitude. It may be repeated that no star of my material, agreeing with the last-mentioned conditions of grouping, has been neglected.

For the hypothesis of solar motion towards $l = 71^\circ$ I obtained the results in Table XIII.

Table XIII

Number of Stars	Co-ordinates of the Centre of Group in Parsecs		Mean Distance in Parsecs	Mean Direction of Motion of Group
	X	*Y*		
				°
20	− 25·8	−111·9	114·7	75·5
95	+ 3·2	+ 10·3	10·8	73·7
147	+ 24·0	+ 72·0	75·9	73·8
18	+ 51·1	+166·2	173·9	72·3
30	+144·9	+483·0	504·3	71·1

For the hypothesis of solar motion towards $l = 55°$ I obtained from other star-groups the results in Table XIV.

Table XIV

Number of Stars	Co-ordinates of the Centre of Group in Parsecs		Mean Distance in Parsecs	Mean Direction of Motion of Group
	X	*Y*		
				°
29	− 67·2	− 80·4	104·8	58·1
81	+ 6·8	+ 10·3	12·3	57·1
143	+ 43·6	+ 62·1	75·8	56·7
21	+112·2	+141·2	180·4	56·1
35	+279·9	+328·8	431·8	55·1

In both cases it is evident that as we progress in the orbit "preceding" the Sun, more stars show a direction of motion towards smaller values of galactic longitude—and this confirms my supposition.

By interpolation it may be broadly stated that the direction of rotation in the immediate vicinity of the Sun is approximately $l = 57°$, if we regard the hypothesis of solar motion towards $l = 55°$ as being more correct. This example shows again that the direction of solar motion is smaller by 2° than the average direction of the motion of stars in its neighbourhood. So the Sun moves towards the perigalacticum. In the section on distribution of direction of the stellar motion I stated that the difference between the direction of solar motion and the most common direction of the motion of giants is only +1°—provided that the Sun moves towards $l = 55°$; for dwarfs the difference is +7°. It is obvious that the group-centre of all the giants considered does not correspond with the Sun but lies in the

orbit "preceding" the Sun—hence the smaller difference for giants. The dwarfs were mostly stars of large proper motions moving altogether towards the apogalacticum ; hence the larger difference. Considering the curved orbit round the galactic centre, it is possible to estimate its distance, using the value of solar motion towards $l = 55°$; this value is approximately

$$10{,}000 \pm 2{,}000 \text{ parsecs (p.e.)};$$

for solar motion towards $l = 71°$ the value is smaller :

$$8{,}000 \pm 1{,}400 \text{ parsecs (p.e.)}.$$

IX. *Oort's Rotational Effect*

14. Having determined that distances from the galactic plane in both—positive or negative—directions do not influence the stellar motions, it was possible to investigate Oort's rotational effect by means of all stars employed —1678 in number. For stars more distant from the galactic plane, their reduced distances from the Sun in the galactic plane have been stated, *i.e.* lengths of their distance-projections on the galactic plane. The direction to the galactic centre has been chosen towards $l = 341°$ and $l = 325°$ respectively, according to the two possibilities of solar motion. Owing to the considerable distance of the galactic centre and to the proportionately small dimensions of the space in which our stars are contained, this space has been divided by lines perpendicular to the first or the second direction : galactic centre—Sun. The distance of a star from the Sun, measured in the direction towards the galactic centre, is

$$R \cos b \cos (l - l_0),$$

where l is the galactic longitude of the star and l_0 the galactic longitude of the centre, R the real distance of the star from the Sun and b the galactic latitude.

According to these distances, star-groups have been formed as follows: in the directions $l = 325°$ and $341°$ respectively, and the distances 0–100, 101–200, 201–400, and over 400 parsecs. Similar groups were formed in the opposite directions, *i.e.* towards $l = 145°$ and $161°$ respectively. This division of space is shown in fig. 7.

The mean velocities for each type and group were computed (computations, owing to homogeneity of material, were made only for giants), and the increase of space-velocity of the stars was calculated to be

$+0{\cdot}0177$ km./sec. parsec^{-1}	for direction of solar motion towards $l = 55°$,
$+0{\cdot}0157$ km./sec. parsec^{-1}	for direction of solar motion towards $l = 71°$.

These numbers refer to star-groups with a maximum mean distance of 1123 parsecs.

X. *Ursa Major Stream*

15. I have shown * that the Ursa Major stream is composed of a great number of stars of type A and some of type F. The total number of stars was 96 ; 80·2 per cent. belonged to type A and 19·8 per cent. to type F. Referring to my work, Professor G. Horn d'Arturo † pointed out an interesting stratification of the Ursa Major stream. Both works are based upon the supposition that the Sun moves with a velocity of 20 km./sec. towards $\alpha = 270°$ and $\delta = +30°$.

It will be interesting to deduce the results for this stream, presuming that stars rotate round the galactic centre. For Rasmuson's and my stars of my cited work I obtain :

Number of Stars	Solar motion towards $l = 71°$		Solar motion towards $l = 55°$	
	Rasmuson's stars	°		°
19	$v = 296{\cdot}7 \pm 3{\cdot}1$ km./sec.	$67{\cdot}7 \pm 2{\cdot}5$	$v = 303{\cdot}7 \pm 4{\cdot}0$ km./sec.	$51{\cdot}9 \pm 1{\cdot}8$
	My stars of type A			
59	$v = 297{\cdot}1 \pm 4{\cdot}7$ km./sec.	$68{\cdot}9 \pm 1{\cdot}7$	$v = 302{\cdot}0 \pm 5{\cdot}0$,,	$52{\cdot}9 \pm 1{\cdot}6$
	My stars of type F			
18	$v = 300{\cdot}4 \pm 8{\cdot}9$ km./sec.	$66{\cdot}7 \pm 2{\cdot}9$	$v = 307{\cdot}7 \pm 11{\cdot}4$,,	$51{\cdot}1 \pm 2{\cdot}5$
	Total stream			
96	$v = 297{\cdot}53 \pm 5{\cdot}19$ km./sec.	$68{\cdot}26 \pm 2{\cdot}06$	$v = 303{\cdot}34 \pm 6{\cdot}02$,,	$52{\cdot}36 \pm 1{\cdot}81$

Mean mass of Rasmuson's stars	$2{\cdot}07\,\odot$
,, ,, my stars of the type A	$2{\cdot}06\,\odot$
,, ,, ,, ,, F	$1{\cdot}36\,\odot$

Considering my previous results we may consider the problem of the Ursa Major stream as solved satisfactorily. In my material stars of types A and F (to be precise, stars with masses between 1·0 and 2·5 ⊙) show sensibly higher velocities. At the same time, those stars prefer two directions only. The first direction is given by the direction of the Ursa Major stream, the second by the Taurus group. The Ursa Major stream is nothing but a motion of stars of masses 1·0–2·5 ⊙, moving towards the perigalacticum (owing to their being of equal mass they move in the same direction, in orbits which have the same eccentricity) ; the Taurus group is formed of stars of the same mass as the stars of the Ursa Major stream—but they move towards the apogalacticum. The velocity of stars of the Ursa Major stream therefore is higher than that of stars of the Taurus group.

The Ursa Major group is defined by the apparent high velocity of its stars as well as by the parallelism of the orbits of the single stars. This can probably be explained by the fact that stars of the Ursa Major group are comparatively near to the Sun (max. distance 100 parsecs), and even systematic errors in the determination of the parallaxes could not change

* *Bull. Astron.*, **6**, 147–153, 1930.

† *Public. dell' Osserv. astr. Bologna*, **2**, No. 9, 99–115, 1931.

considerably the determinations of the velocities of those stars according to magnitude and direction.

XI. *Conclusions*

16. Based upon determined space-velocities of stars (supposing, in the first case, the solar motion towards $l = 71°$ and velocity 300 km./sec., and in

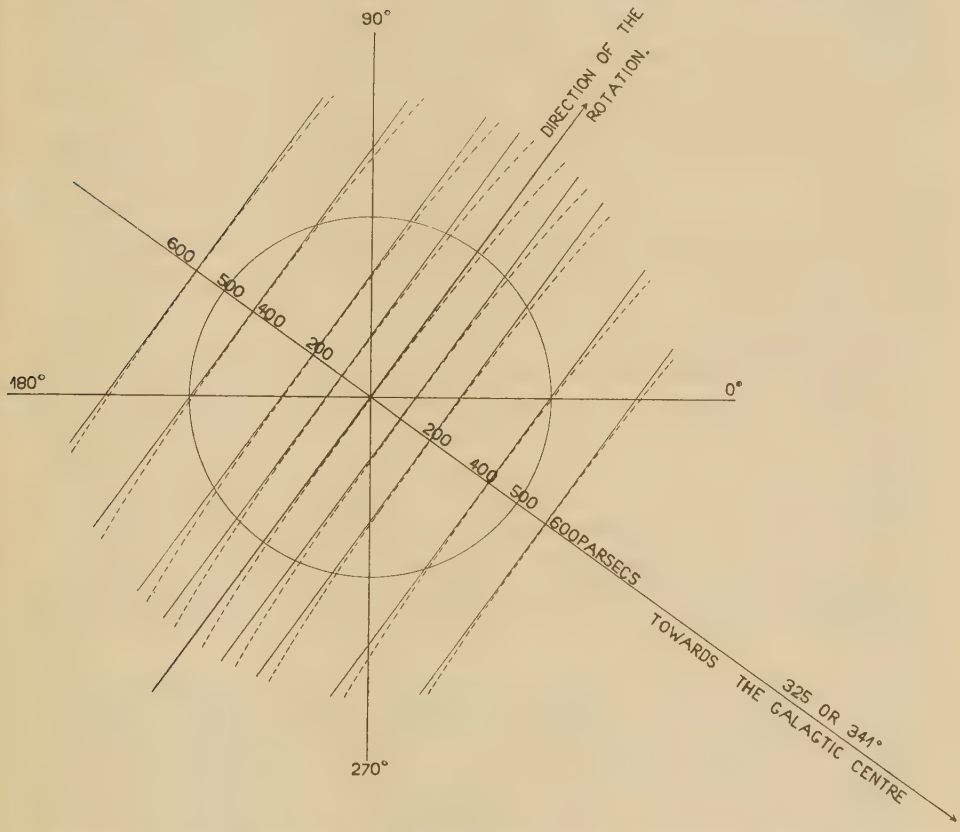


FIG. 7.—*Division of Space for the Determination of Oort's Rotational Term. The curves signify circular orbits about the galactic centre at the distance of* 10,000 *parsecs.*

the second towards $l = 55°$ and the same velocity) and their masses, calculated by Eddington's method, the following results for the rotational movements of stars are obtained :—

(i) The velocity of homogeneous star-groups at any distance within the limits of 1 – 150 parsecs from the galactic plane, in both directions, is constant, and its magnitude gives the solar velocity relative to the star-groups considered. The motion of stars within these limits is thus independent of the distance of the stars from the galactic plane.

(ii) The distribution of stellar velocities is typical for both giants and dwarfs. Curves of velocity-distribution for giants reach a maximum between 280 and 300 km./sec. The most massive stars of spectral types B and M show

a maximum between 280 and 290 km./sec.—less massive stars of types K, G, F and A show their maxima towards higher values. From these maxima we see a sudden falling towards high velocities, so that velocities over 340 km./sec. are scarcely to be found, in spite of the theoretical limiting velocity of approximately 420 km./sec.

The maximum distribution of the velocities for dwarfs is between 250 and 320 km./sec., and the curves do not show any visible falling towards high velocities. An analysis of the space-velocities of the nearest dwarfs (in a sphere of radius 10 parsecs) shows, however, that this distribution does not accord with reality—because the mean velocities of dwarfs are in reality higher than those of giants.

(iii) There is a linear relation between the velocities and the masses of stars, showing that with decreasing mass the space-velocity of stars increases. The analysis of the present material shows this relation very clearly in giants ; dwarfs show a sensible falling of stellar velocity from the mass 1·0 ⊙. Type A stars, with masses 1·4–2·4 ⊙, move with the highest velocity round the galactic centre.

(iv) Discussion of the proper motions shows, however, that 93·2 per cent. of the 518 dwarfs here employed have proper motions μ_α, $\mu_\delta > \pm 0''{\cdot}100$. Both giants and dwarfs show that the higher their proper motions, the smaller are their space-velocities. This is very important for dwarfs, because dwarfs with proper motions μ_α, $\mu_\delta < \pm 0''{\cdot}100$ are practically absent from my material. If we had those stars in our material, then we should see that mean space-velocities of dwarfs are higher than those of giants. The real relation between mass and space-velocity is given most probably for dwarfs also as follows : The rotational velocity of stars with small mass is greater than the rotational velocity of stars with greater mass.

(v) Taking into account the magnitude of the proper motions of dwarfs, we may wonder why they show mostly small velocities. The explanation is as follows : These stars move at the same time towards higher values of the angle α (galactic longitude), and it seems as though these dwarfs are pushed out of rotation. As stars moving at the apogalacticum can move towards higher angle-values, so it is clear also that their velocities are smaller. Dwarfs, again, which might move towards the perigalacticum—towards smaller values of the angle α and with higher velocity—are not represented in our material owing to their proper motions being necessarily small. This also accounts for the different directions of the major axis of the velocity-ellipsoid for giants and dwarfs.

(vi) The fact that smaller masses (those with smaller radius) move round the galactic centre with higher velocities than larger masses may be due to the resistance of interstellar matter.

(vii) The mean direction of motion of the stars here employed is given by the galactic longitude value of 56° for giants and 62° for dwarfs—hence we can conclude that the Sun moves towards a smaller galactic longitude than that of the mean direction of motion of stars in the immediate vicinity of the Sun. The difference between solar motion and the mean rotational

direction of stars in the immediate neighbourhood of the Sun is necessarily higher than the difference between solar motion and the mean direction of motion of giants, if we take into consideration that the space-distribution of stars here employed is not homogeneous in all directions of galactic longitude. The regions between $340 - 90 - 180°$ of galactic longitude contain about 80 per cent. of the stars, and (with regard to stellar rotation) the mean direction of their motion is smaller. Taking account of this circumstance I obtained the following result: The mean motion of stars in the vicinity of the Sun is given by a direction which is 2° higher (in longitude) than the galactic longitude of solar motion.

The fact that the solar velocity of 300 km./sec. is higher than the average velocity of other stars is explained not only by the consideration that the Sun is a dwarf but also by reason of its motion (in its elliptic orbit round the galactic centre) towards perigalacticum.

(viii) An analysis of stellar velocities according to their mean direction of motion proves that stars move in elliptic orbits round the galactic centre, and that velocities towards galactic longitudes $l < 57°$ must necessarily be higher than velocities towards the values $l > 57°$. This dependence is given (for the theory of solar motion towards $l = 55°$) for all stars by the following values of the mean direction of motion:—

km./sec. <270	271–280	281–290	291–300	301–310	311–320	km./sec. >320
$59°{\cdot}48_{(321)}$*	$57°{\cdot}56_{(248)}$	$57°{\cdot}46_{(375)}$	$56°{\cdot}54_{(363)}$	$54°{\cdot}19_{(242)}$	$56°{\cdot}81_{(85)}$	$48°{\cdot}65_{(34)}$

For the second theory ($l = 71°$) the relation is not the same—from which it may be concluded that the Sun moves towards $l = 55°$ or, better, towards $l = 53°$.

(ix) Higher values for the length of the major axis of the velocity-ellipsoid for dwarfs (that is, higher dispersion of velocity with regard to direction) correspond to higher eccentricity of the orbits of dwarfs round the galactic centre. This higher eccentricity determines also considerable differences between the velocities of dwarfs which occur within the limits 250–320 km./sec., as has been well demonstrated.

(x) The direction of the major axis of the velocity-ellipsoid of giants towards higher values of galactic longitudes is known. This is due to the large number of distant stars which are in the orbit preceding the Sun. Their mean direction of motion is given by a smaller galactic longitude than that of stars in the immediate locality of the Sun. Consequently, the major axis of the velocity-ellipsoids must tend towards higher values of the galactic longitude—that is, approximately towards $l = 340°$.

The reverse is the case for dwarfs. They are found in the vicinity of the Sun. Most of these stars, having large proper motions, move with low velocity towards higher values of galactic longitudes. In the direction towards smaller values of the angle, only stars having small proper motions would move—and these stars are not represented in the material of dwarfs.

* Numbers in brackets indicate the number of stars.

Such stars would have high velocities and would move towards the perigalacticum. Owing to the large number of slow dwarfs tending towards the apogalacticum, and to the lack of quick stars moving to the perigalacticum, the major axis of the velocity-ellipsoid inclines to lower values of the galactic longitudes—approximately towards $l = 320°$. Hence may be concluded that the real value of the galactic centre is somewhat higher, say $l = 325°$.

(xi) A new method for the demonstration of stellar rotation is given, based on the determination of the direction of rotation at different distances from the Sun, measured in, and opposite to, the direction of rotation. From the determined curvature of the direction of stellar motion the distance of the rotational centre is estimated to be

$$10{,}000 \pm 2{,}000 \text{ parsecs (p.e.).}$$

(xii) The value of Oort's rotational effect determined from my material is

$$+0{\cdot}0177 \text{ km./sec. parsec}^{-1}.$$

(xiii) The mean mass of the stars in the neighbourhood of the Sun is given by the value

$$0{\cdot}79\odot \pm 0{\cdot}36 \text{ (p.e.).}$$

This value forms the superior limit of the estimation. It will be somewhat smaller in reality because of the fact, already mentioned, that not all dwarfs within a sphere of 10 parsecs are exhausted by the number 76.

(xiv) The Ursa Major stream is a stream of stars having the mean mass $1{\cdot}4$–$2{\cdot}4\odot$. Therefore it is formed mainly of stars of the types A and F. Stars of the type F belonging to that stream show higher velocities than stars of the same stream of the type A—corresponding very well with the rule determined by me, that rotational stellar velocity is higher as the mass is smaller. Actually, the mass of stars of this stream belonging to the type F is only $1{\cdot}4$ of the Sun's mass, and for the stars of the type A the mass is $2{\cdot}1$. So that stars of the Ursa Major stream of type F belong rather to dwarfs than to giants.

The mean direction of the motion of this stream is given by the galactic longitude, 52° approximately. The Ursa Major stream consists of stars approaching the perigalacticum, and therefore the mean velocities of the Ursa Major group are higher than those of all stars of the type A or of dwarfs of the type F.

(xv) The real velocity of a star depends on two factors: the first is the mass, the second is whether the star moves towards the perigalacticum or towards the apogalacticum. The direction of this velocity relative to the Sun is given by the position of a star in the orbit round the galactic centre and its motion towards the peri- or apo-galacticum. It is a characteristic fact that giants move in more circular orbits, and dwarfs in elliptic orbits with considerable eccentricity.

(xvi) All values determined in galactic co-ordinates are computed according to the position of the galactic pole given by Kapteyn : $\alpha = 190°{\cdot}6$ and $\delta = +27°{\cdot}2$. The position of the x-axis is the intersection of the galactic plane with the equator.

It gives me pleasure to thank Professor W. W. Heinrich, of the Charles University at Prague, for his valuable advice. Also I desire to express my thanks to my wife for the valuable help she has given me in numerical computations, and to Miss Violet White for the care which she has taken in putting the paper into its final form.

Comenius University (Physical Department),
Bratislava:
1931 *November.*

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT TOMSK UNIVERSITY OBSERVATORY DURING THE YEAR 1931.

N. Goryatscheff.

($\phi = 56° 28' 07''$; $l = 5^h 39^m 47^s{\cdot}4$ east of Greenwich)

Date (U.T.)	Local Sidereal Time	Star	Mag.	Phase	Δt
1931.	h m s				s
II 22·572891	5 30 45·2	B.D. +10° 225	8·3	D	+13·5
II 24·646668	7 25 10·1	τ Arietis	5·1	D	+ 9·1
II 25·587071	6 03 03·4	B.D. +24° 633	8·5	D	+ 6·3
II 25·717216	9 11 00·7	B.D. +24° 642	8·7	D	+11·5
II 25·724219	9 21 05·5	B.D. +24° 643	8·2	D	+ 7·5
II 28·676965	8 24 41·2	B.D. +27° 1362	6·9	D	+ 7·7
III 25·596548	8 07 08·0	B.D. +26° 759	7·8	D	+ 9·6
III 28·720854	11 18 27·1	ω Cancri	6·1	D	+ 7·1
III 31·568827	7 50 45·7	l Leonis	5·2	D	+ 8·5
IV 21·624499	10 33 56·5	B.D. +25° 723	8·4	D	+ 7·2
IV 21·709713	12 36 59·1	B.D. +25° 731	7·5	D	+ 6·9
IV 22·636930	10 55 50·0	B.D. +27° 828	9·0	D	+ 7·7
IV 22·653694	11 20 02·4	B.D. +27° 833	8·2	D	+ 9·8
IV 22·660765	11 30 15·0	B.D. +27° 832	8·5	D	+ 1·0
IV 22·665365	11 36 53·5	B.D. +27° 837	8·8	D	+10·3
IV 22·676918	11 53 34·5	B.D. +27° 839	8·5	D	+ 8·3
IV 22·713893	12 46 57·8	B.D. +27° 849	7·8	D	+ 6·4
IV 22·764729	14 00 22·1	B.D. +27° 866	8·0	D	+10·6
IV 23·713430	12 50 14·3	B.D. +27° 1212	8·2	D	+ 6·1
IV 23·737974	13 25 40·7	B.D. +27° 1219	6·8	D	+ 7·9
IV 24·659675	11 36 33·8	B.D. +26° 1625	8·6	D	+10·1
IV 24·708040	12 46 23·9	c Geminorum	5·5	D	+ 9·2
V 20·702478	14 20 52·5	B.D. +27° 1129	8·7	D	+ 7·7
V 21·667474	13 34 16·4	B.D. +26° 1564	7·2	D	+11·3
V 21·667544	13 34 22·5	B.D. +26° 1563	8·9	D	+ 8·3
V 23·686790	14 10 03·0	B.D. +19° 2209	8·8	D	+ 6·5

Apparent Places of Occulted Stars of Magnitude >6·5

Star B.D.	α	δ	Authority
°	h m s	° ′ ″	
+10 225	1 39 09·57	+10 44 21·5	Carte Phot. (Toulouse), +11° : 140 : 126.
+24 633	4 07 52·18	+24 23 39·7	Abbadia 1933 ; Carte Phot. (Paris), +24° : 408 : 15.
+24 642	4 13 59·91	+24 44 44·1	Abbadia 1980 ; Carte Phot. (Paris), +24° : 408 : 52 ; 416 : 8
+24 643	4 14 17·33	+24 50 20·1	Abbadia 1980 ; Carte Phot. (Paris), +24° : 408 : 53 ; 416 : 11.
+27 1362	7 15 16·04	+26 57 08·8	Gr_{1910} 3714.
+26 759	4 48 41·68	+26 40 02·7	Gr_{1910} 2070.
+25 723	4 36 53·28	+25 51 44·3	Gr_{1910} 2006.
+25 731	4 41 56·93	+25 54 49·7	Gr_{1910} 2027.
+27 828	5 35 23·72	+27 44 47·7	Gr_{1910} 2480.
+27 833	5 36 19·78	+27 45 09·6	Gr_{1910} 2495.
+27 832	5 36 12·49	+27 24 34·6	Gr_{1910} 2492.
+27 837	5 37 06·64	+27 36 19·3	Gr_{1910} 2505.
+27 839	5 37 42·01	+27 44 24·2	Gr_{1910} 2519.
+27 849	5 40 00·56	+27 42 16·4	Gr_{1910} 2553.
+27 866	5 43 24·86	+27 36 52·8	Gr_{1910} 2604.
+27 1212	6 39 53·02	+27 33 10·6	Gr_{1910} 3315.
+27 1219	6 41 15·69	+27 44 54·5	Gr_{1910} 3334.
+26 1625	7 37 05·24	+26 03 09·3	Gr_{1910} 3922.
+27 1129	6 25 24·17	+27 40 28·5	Gr_{1910} 3133.
+26 1564	7 23 46·76	+26 22 14·6	Gr_{1910} 3787.
+26 1563	7 23 40·88	+26 19 07·7	Ox ph. +26°·20856. +27°·18873.
+19 2209	9 21 12·22	+19 02 54·9	Abbadia 5549; Carte Phot. (Paris), +20° : 920 : 128.

The proper motions are computed from Schorr, *Eigenbewegungslexikon.*

Observer : Prof. N. Goryatscheff.

Instruments : Zeiss telescope, 90-mm. aperture, magnifying power 90. Observations made with eye and ear.

The time-corrections of the chronometer were found from the Bordeaux, Rugby and Nauen wireless signals.

The Partial Eclipse of the Sun

Date (U.T.)	Local Sidereal Time	Phase
IV 18·058998	$20^h\ 45^m\ 33^s·77$	2nd contact

Observer : Prof. N. Goryatscheff.

Tomsk, Siberia, U.S.S.R.,
University Astronomical Observatory:
1932 *March* 10.